Fast computation of higher dimensional isogenies for cryptographic applications

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Introduction

Hard problems for public key cryptography

Widely used underlying problems...

- The factorisation problem (RSA):

$$N = p \times q.$$

- The discrete logarithm problem in a group $(G, +)$ generated by P (ECC):

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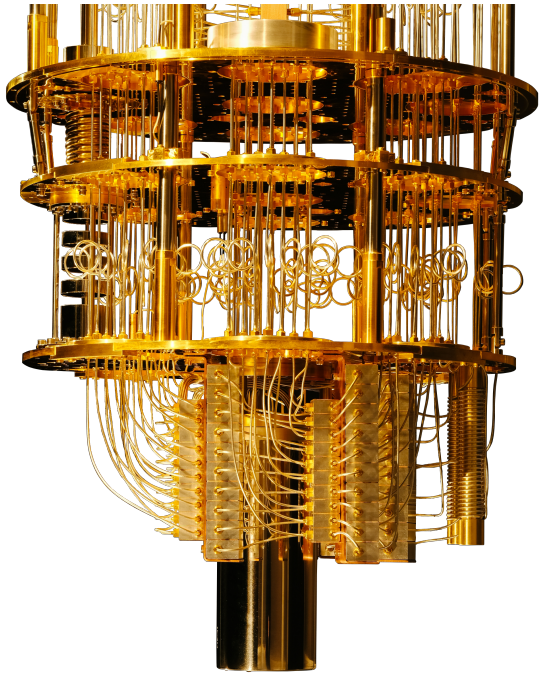
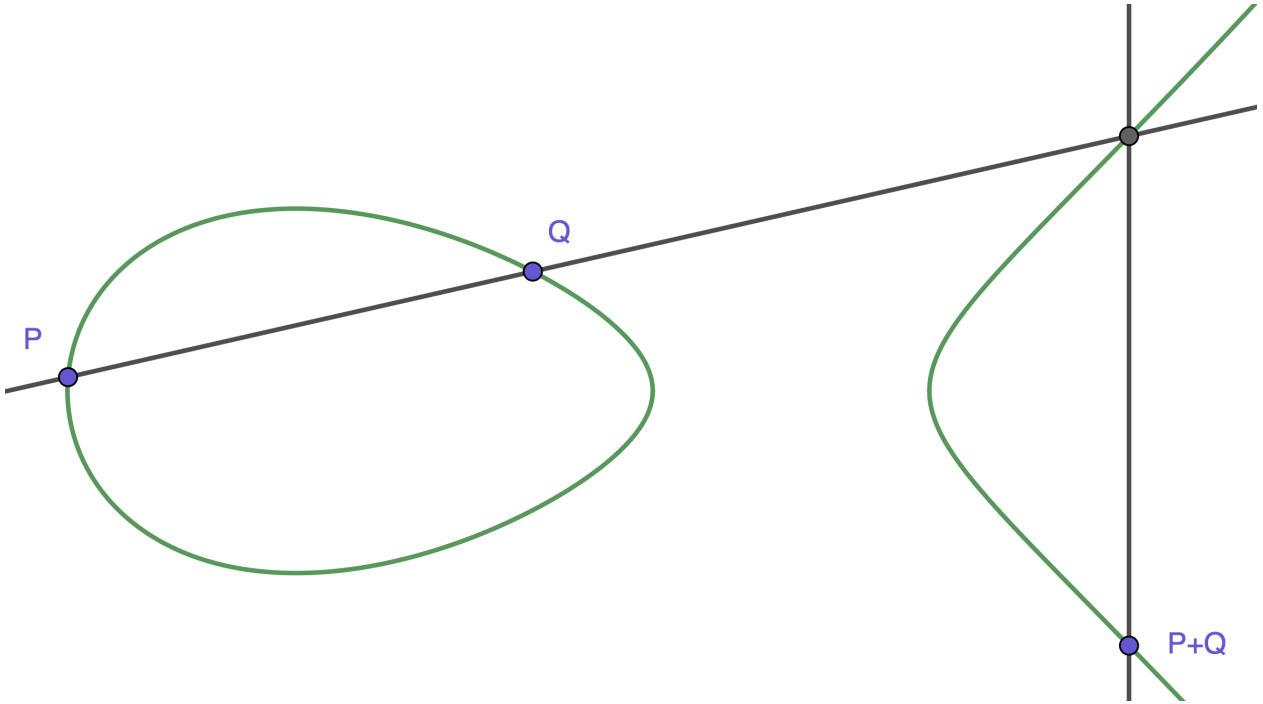
...All broken by quantum computers



Elliptic curves

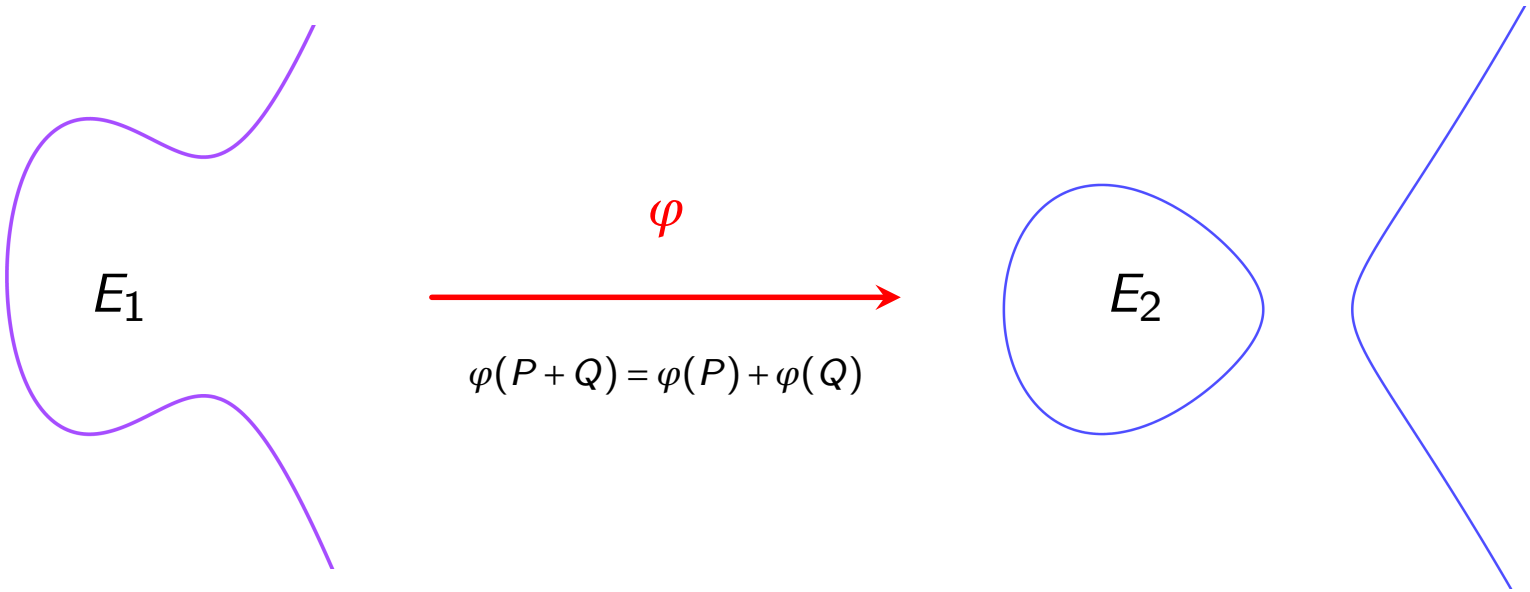


- An elliptic curve $E/\mathbb{F}_q$ is defined by:

$$y^2 = x^3 + ax + b, \quad a, b \in \mathbb{F}_q$$

  with an infinite element 0_E.
- E is equipped with a commutative group law.

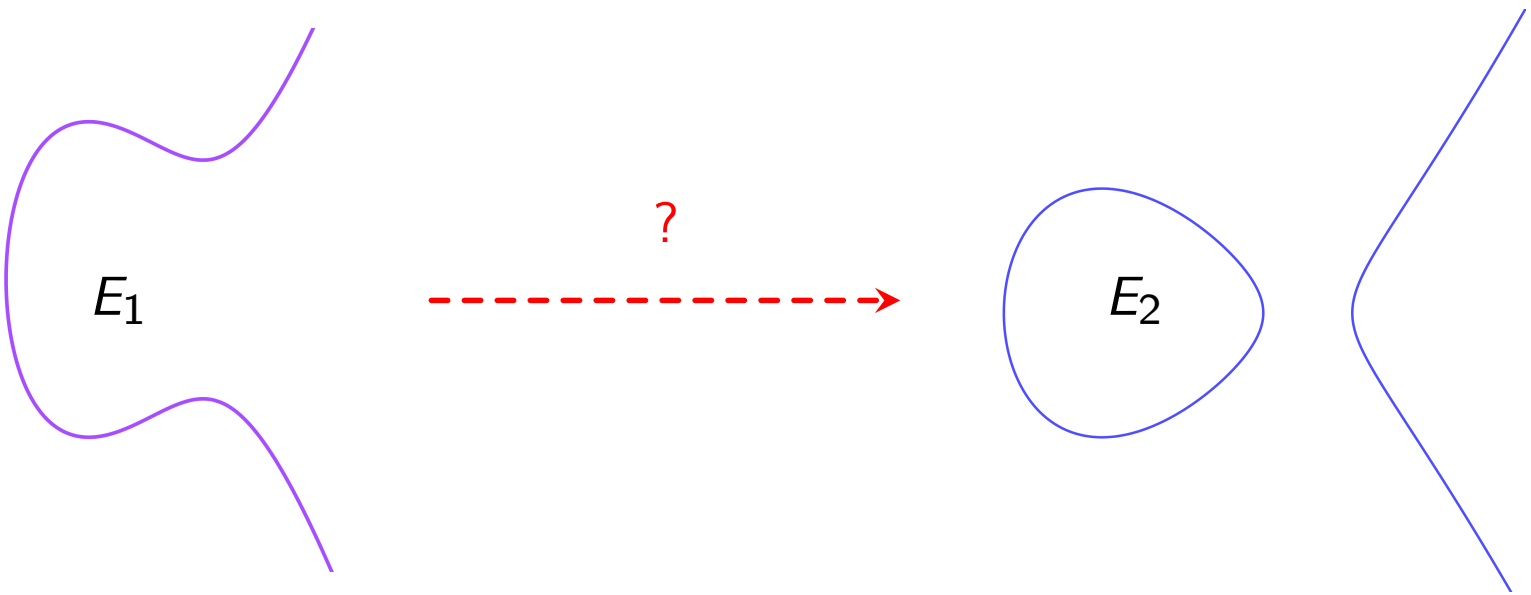
Isogenies



$$\varphi(x,y)=\left(\frac{p(x)}{q(x)},y\frac{r(x)}{s(x)}\right)$$

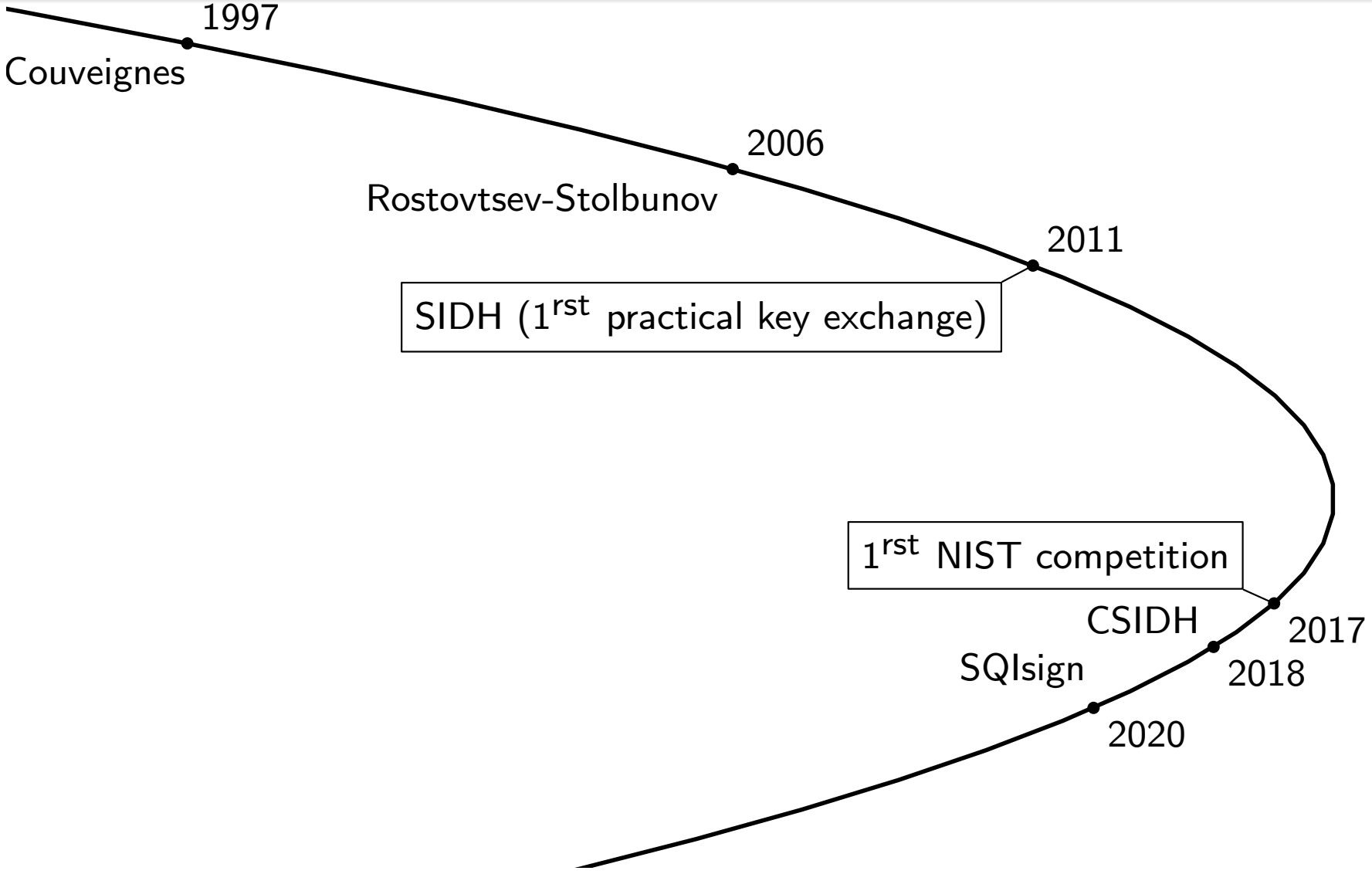
Why are isogenies interesting in cryptography?

The isogeny problem: Given two elliptic curves $E_1, E_2/\mathbb{F}_q$, find an isogeny $E_1 \longrightarrow E_2$.



This problem is assumed to be hard for both classical and quantum computers.

A brief (biased) history of isogeny based cryptography



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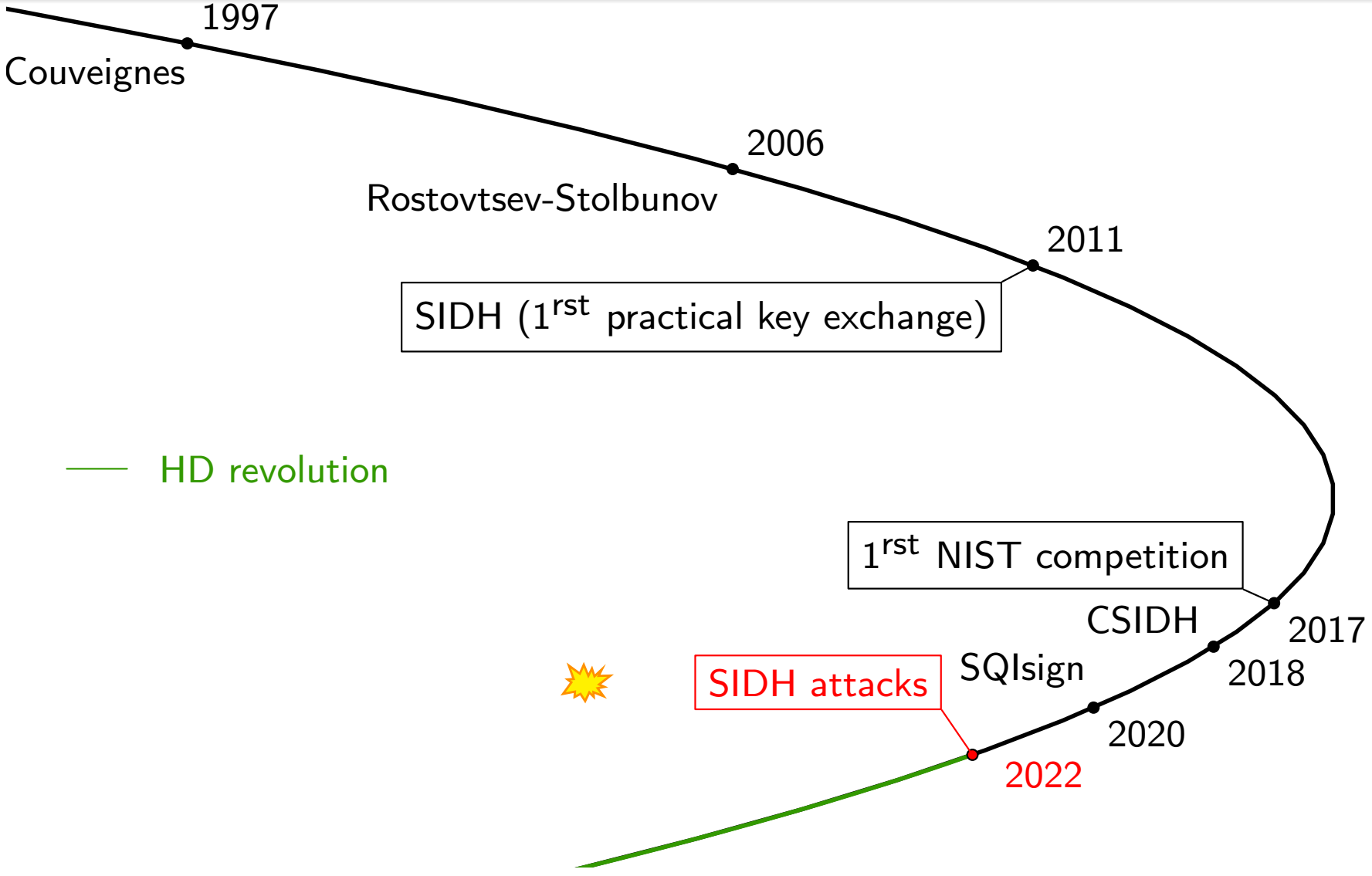

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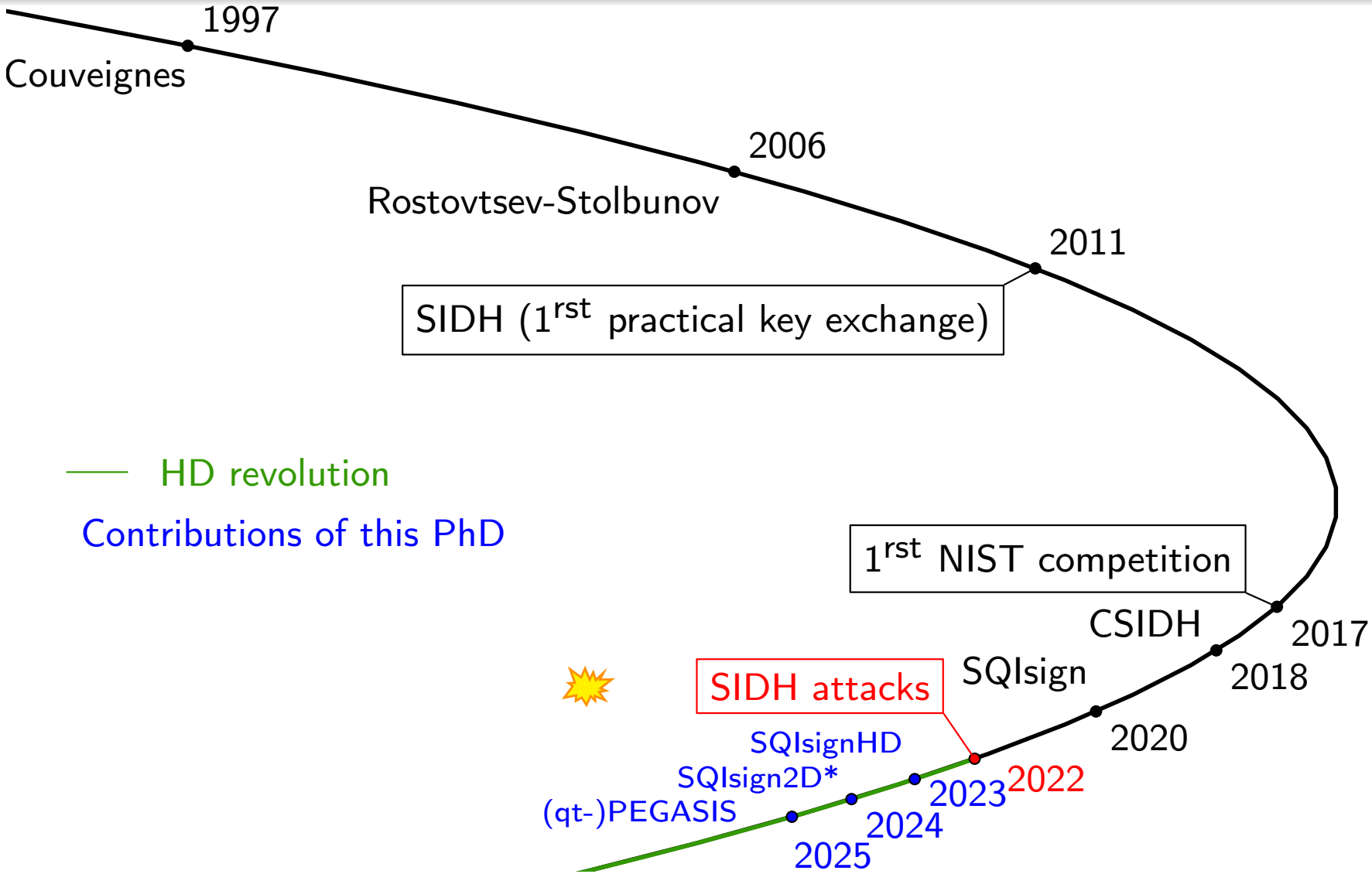
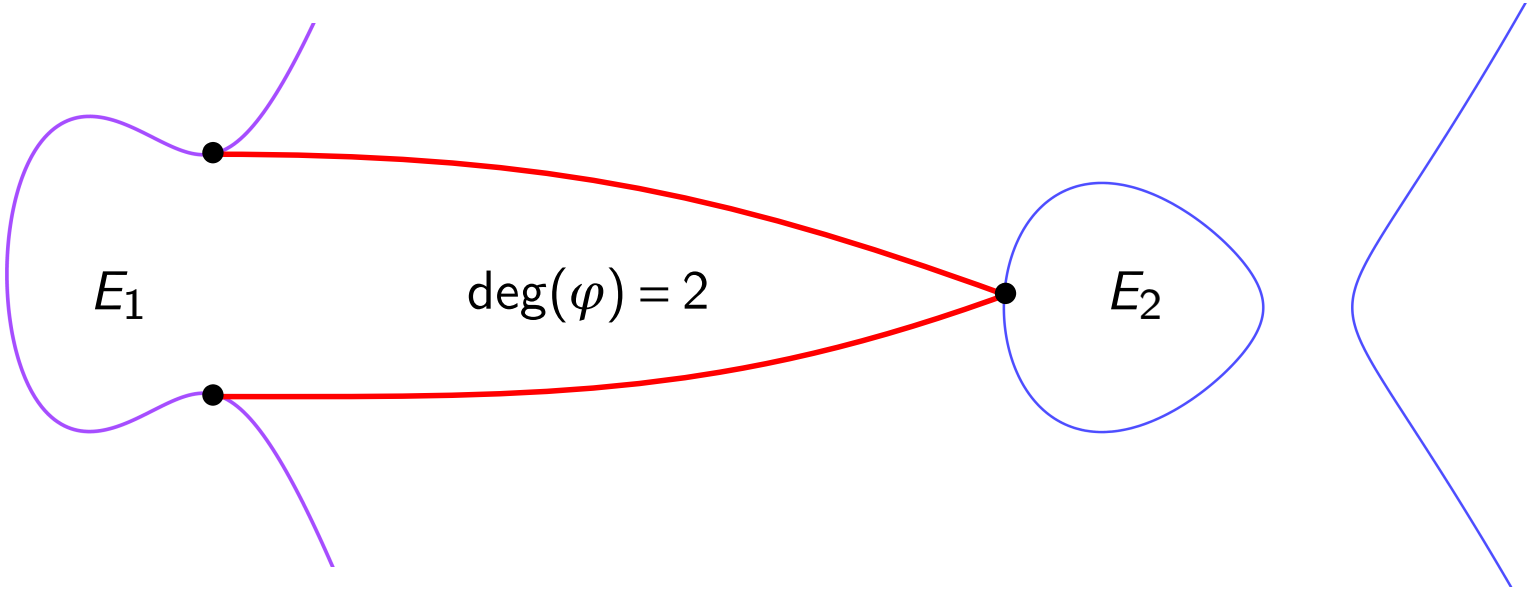


Isogenies - degree

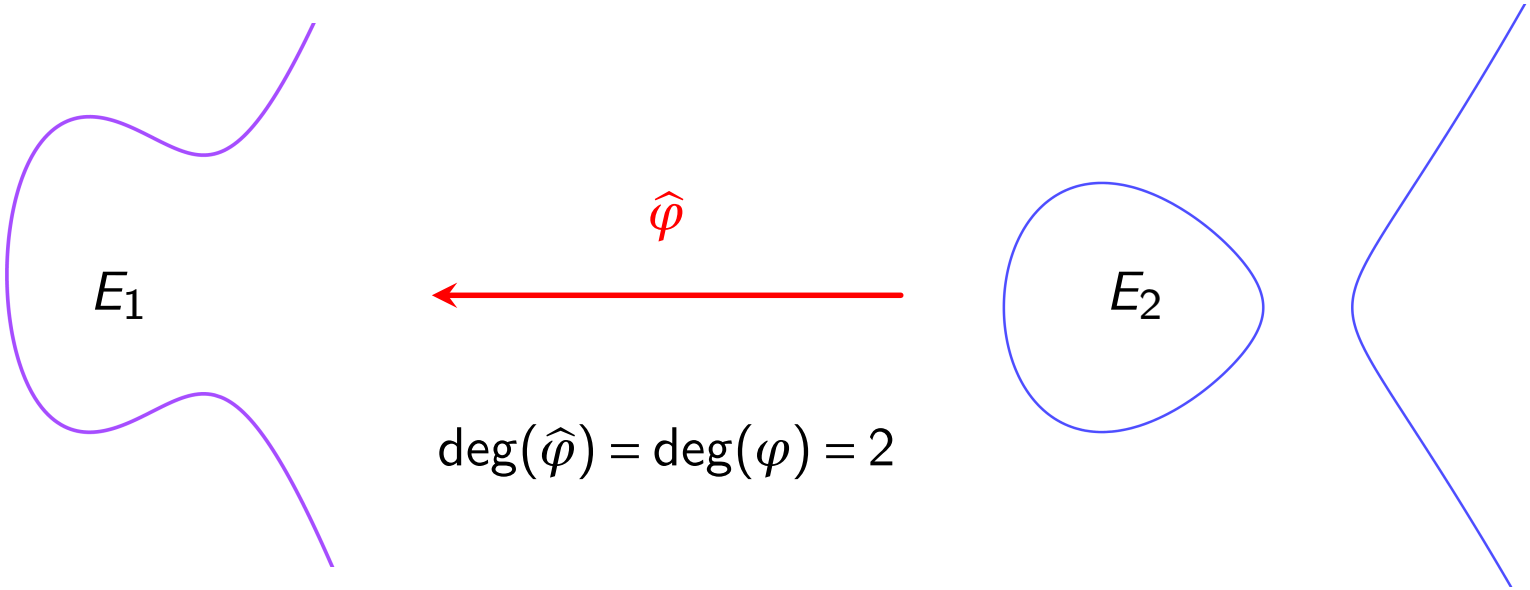


- The "size" of an isogeny: $\deg(\varphi) = \max(\deg(p), \deg(q))$ when

$$\varphi(x, y) = \left(\frac{p(x)}{q(x)}, y\frac{r(x)}{s(x)}\right)$$

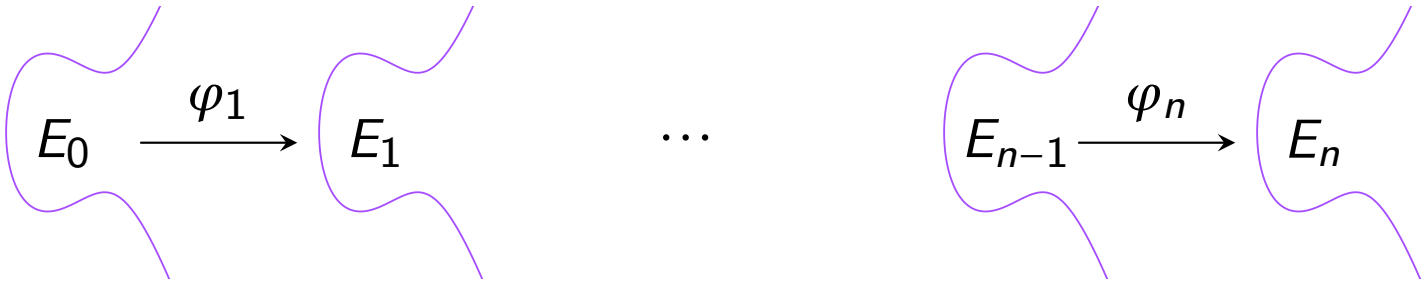
- $\deg(\varphi) = \#\ker(\varphi)$ in most cases.

Isogenies - the dual isogeny



If $\deg(\varphi) = n$, then $\widehat{\varphi} \circ \varphi = [n]$.

Isogeny chains



$$\deg(\varphi_n \circ \cdots \circ \varphi_1) = \prod_{i=1}^{n} \deg(\varphi_i)$$

How to efficiently represent an isogeny of big degree

✗ Inefficient: with rational fractions

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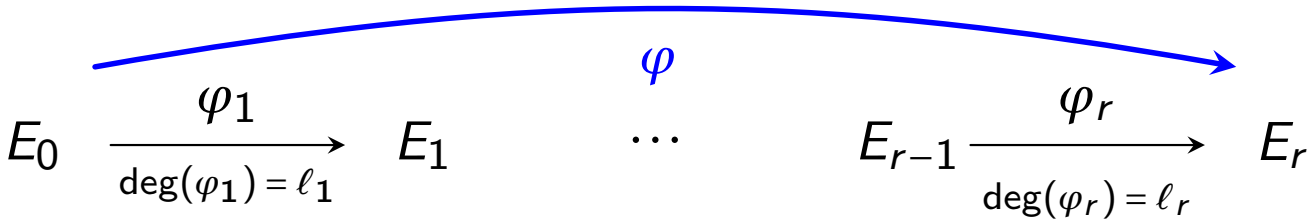

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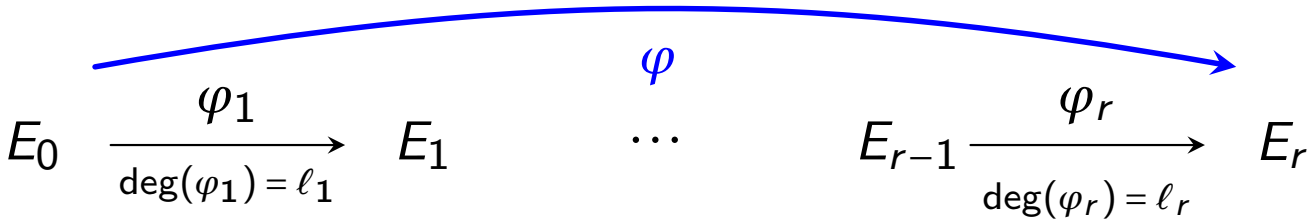

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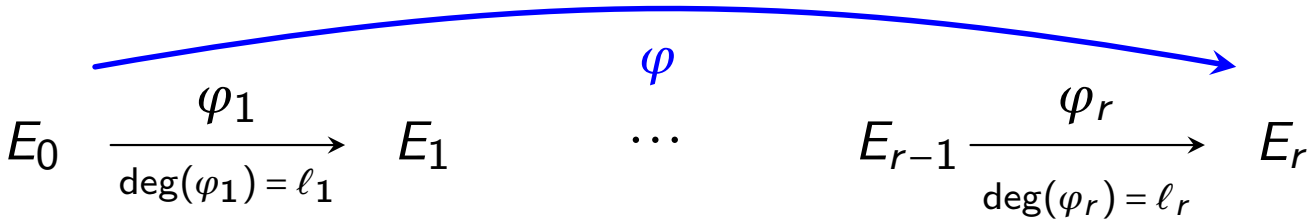

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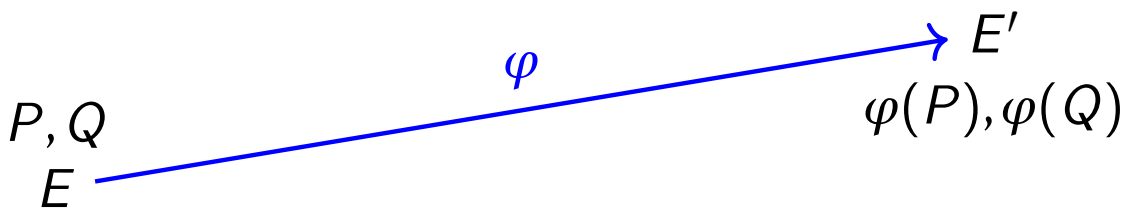
✓ **New:** If $\deg(\varphi)$ is not smooth, with higher dimensional interpolation



SQIsign and the Deuring correspondence

The Endomorphism ring

Definition (Endomorphism ring)

$$\mathrm{End}(E) = \{0\} \cup \{\text{Isogenies } \varphi : E \longrightarrow E\}$$

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Let $E/\mathbb{F}_q$ $(p = \mathrm{char}(\mathbb{F}_q))$. *Then* $\mathrm{End}(E)$ *is either isomorphic to:*

- *An order in a quadratic imaginary field (2D commutative object). We say that* E *is ordinary.*
- *An order in the quaternion algebra* $\mathcal{B}_{p,\infty}$ *(4D non-commutative object). We say that* E *is supersingular.*

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In isogeny-based cryptography, supersingular curves are prefered for security and efficiency reasons.

The Deuring correspondence

Supersingular elliptic curves	**Quaternions**
$j(E)$ or $j(E)^p$ supersingular	$\mathcal{O} \cong \mathrm{End}(E)$ maximal order in $\mathcal{B}_{p,\infty}$

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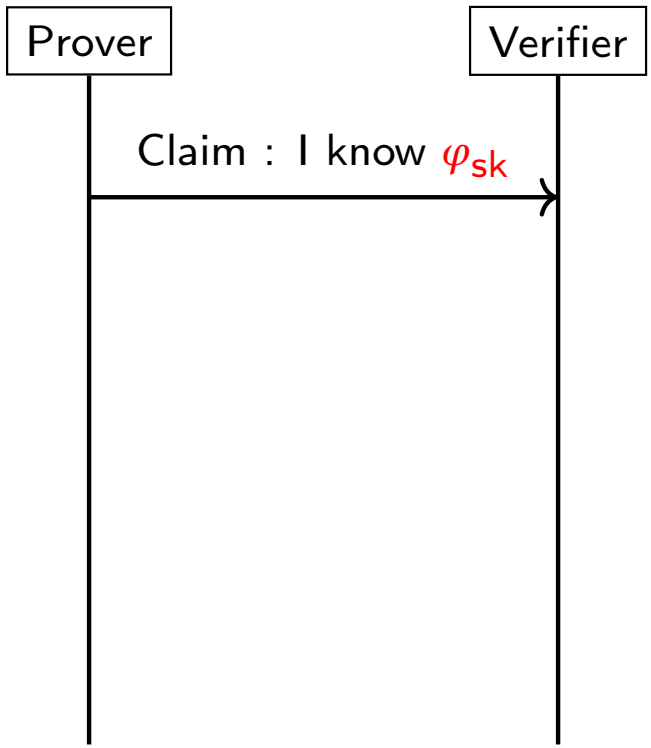
$\deg(\varphi)$	$\mathrm{nrd}(I_\varphi)$

The SQIsign identification scheme



public

Prover's secret

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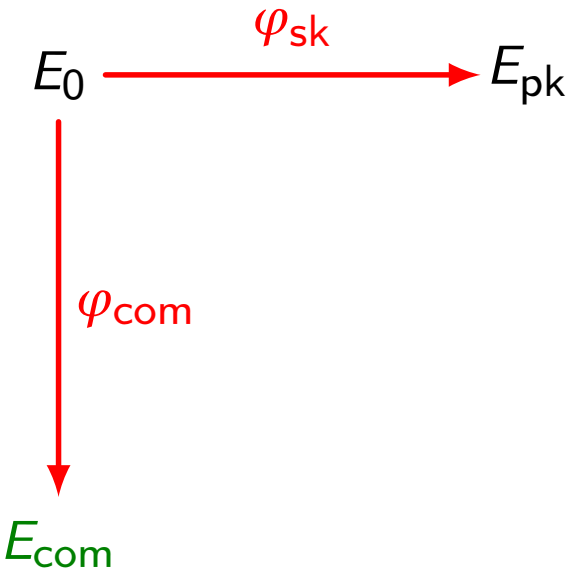

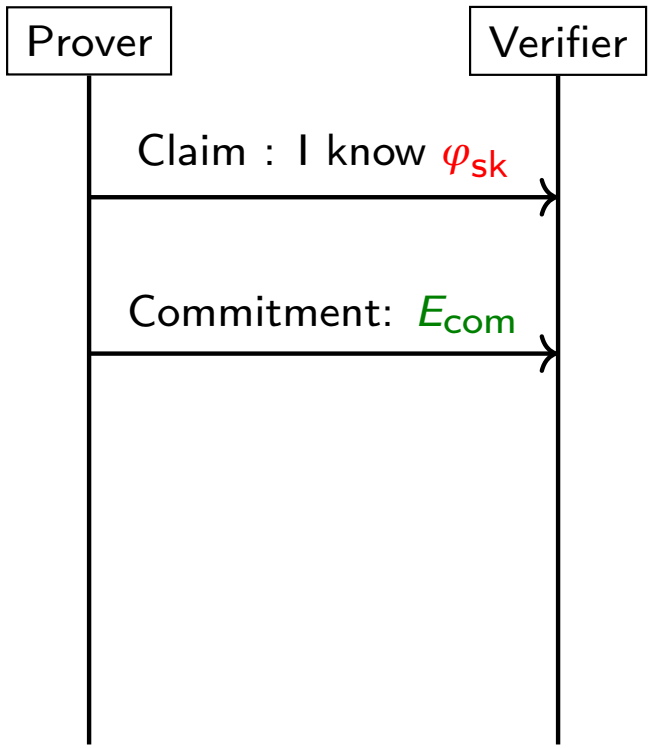

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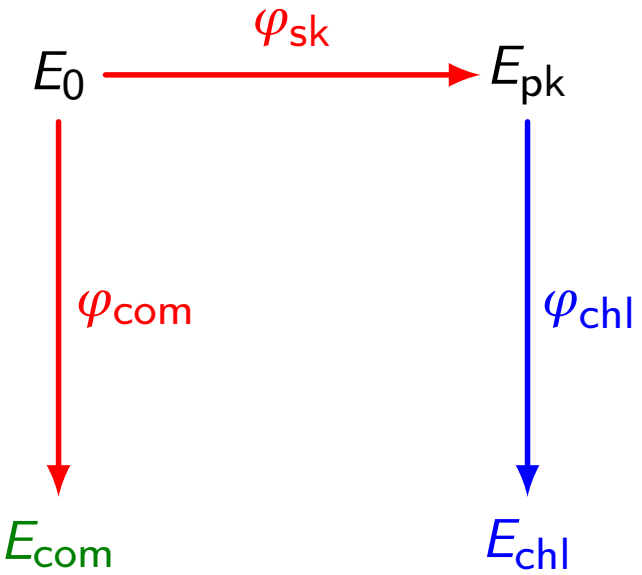

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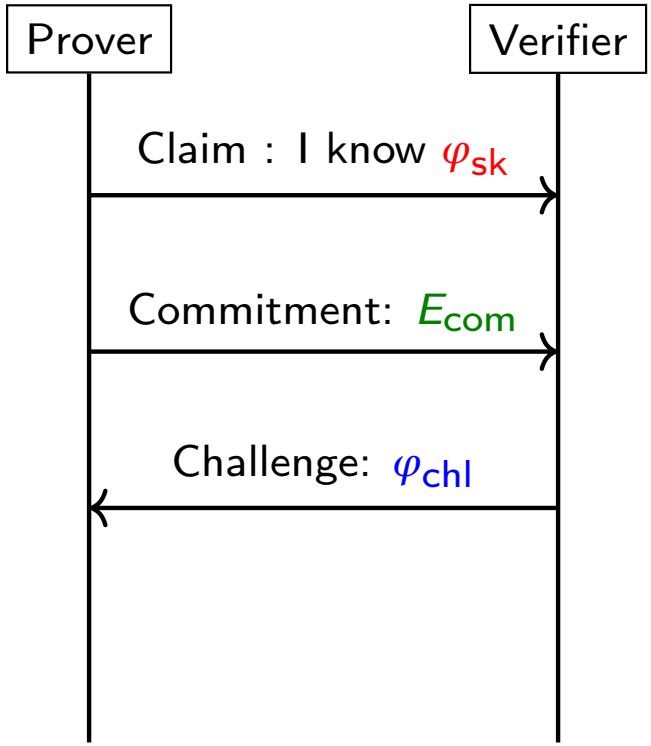

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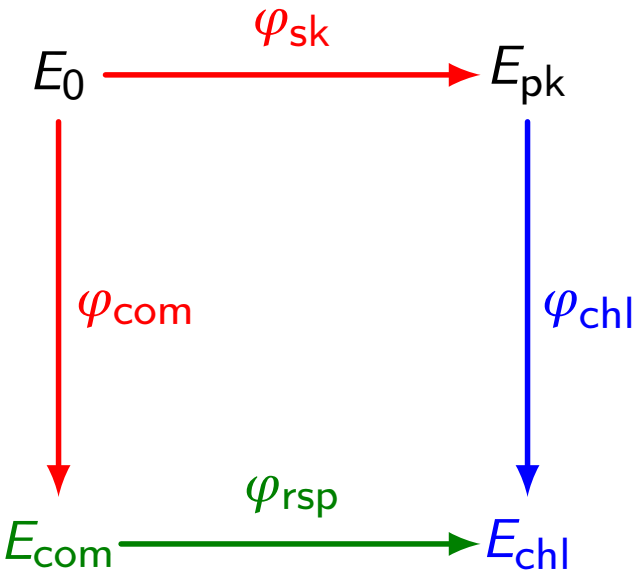

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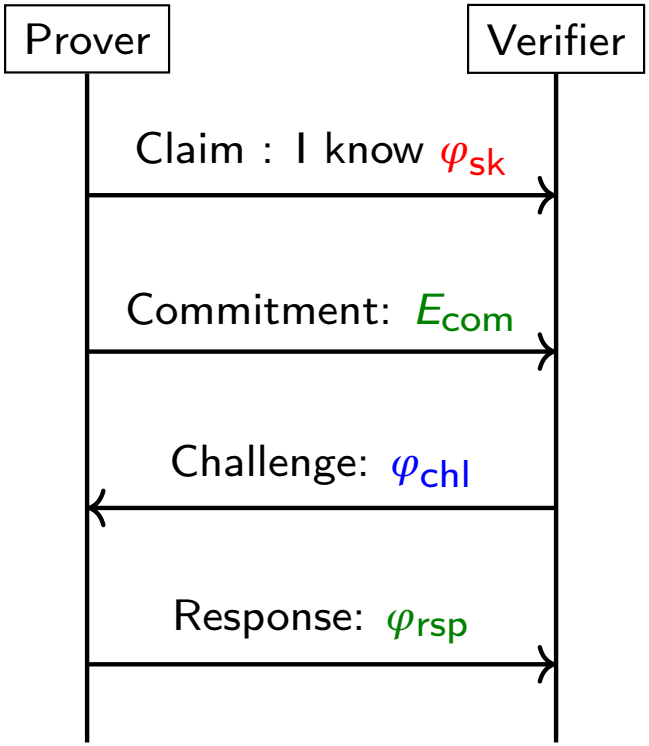

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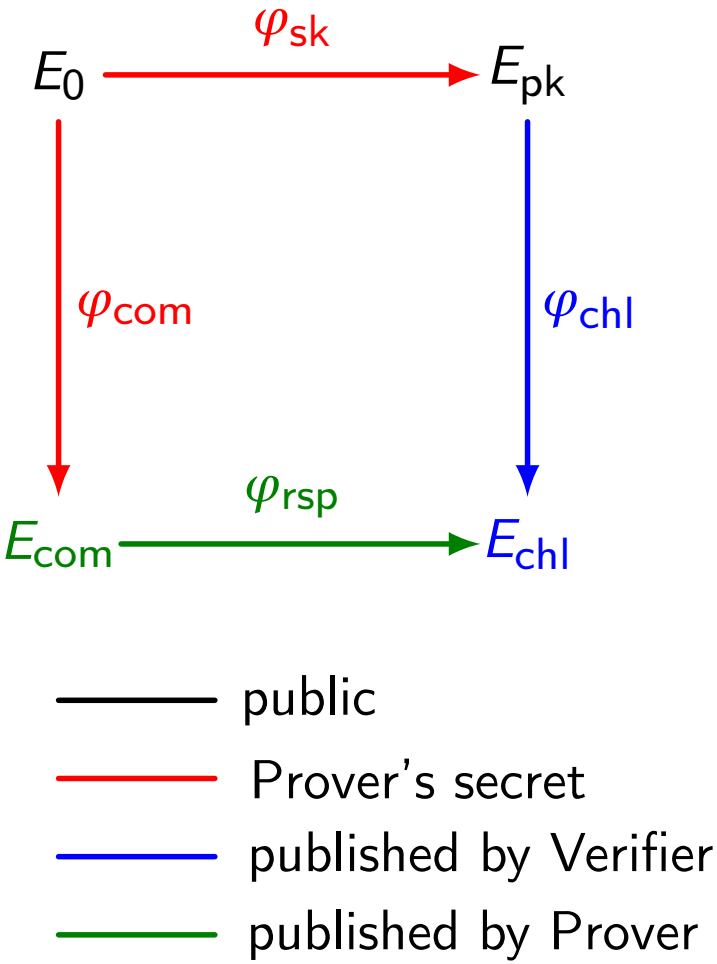

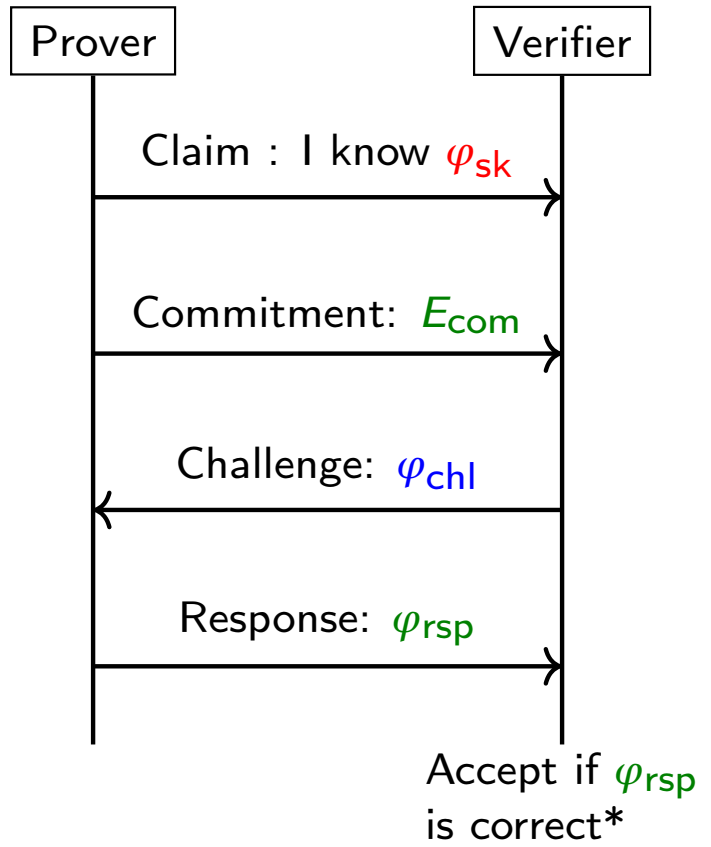

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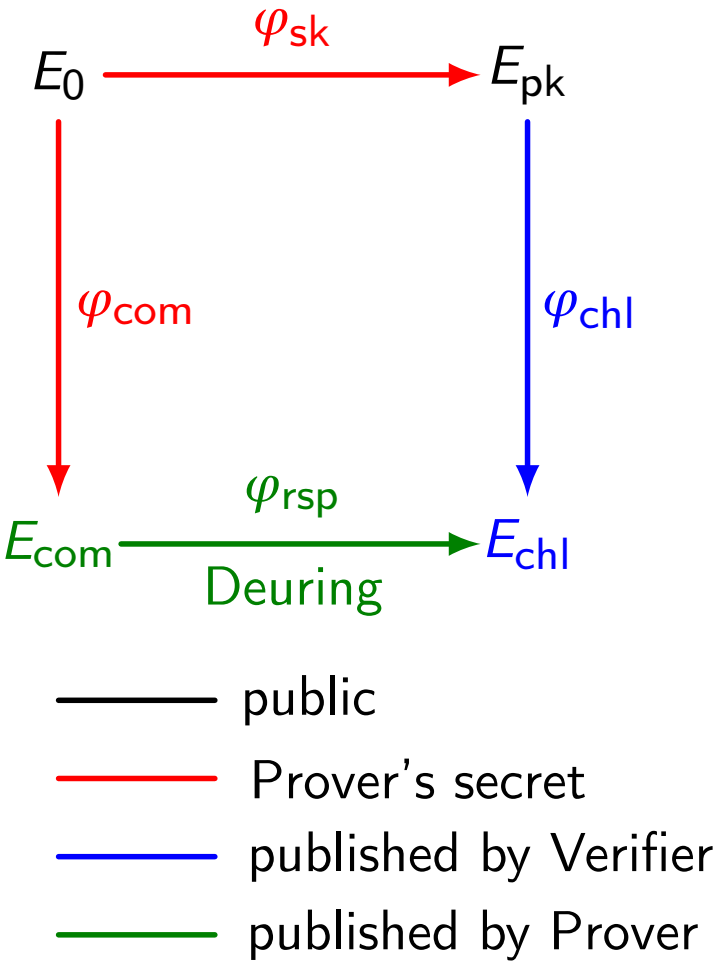

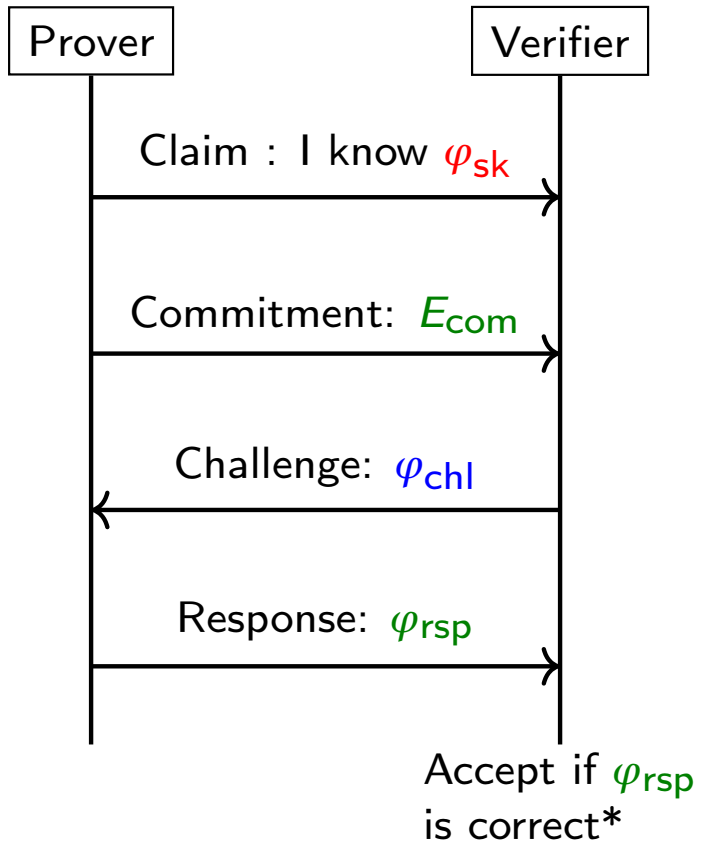

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Computing isogenies via the Deuring correspondence

Goal: In SQIsign, we know $\mathrm{End}(E_{\mathsf{com}})$ and $\mathrm{End}(E_{\mathsf{chl}})$ and we want an isogeny $\varphi_{\mathsf{rsp}} : E_{\mathsf{com}} \longrightarrow E_{\mathsf{chl}}$.

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⚠ Efficiency of the translation.

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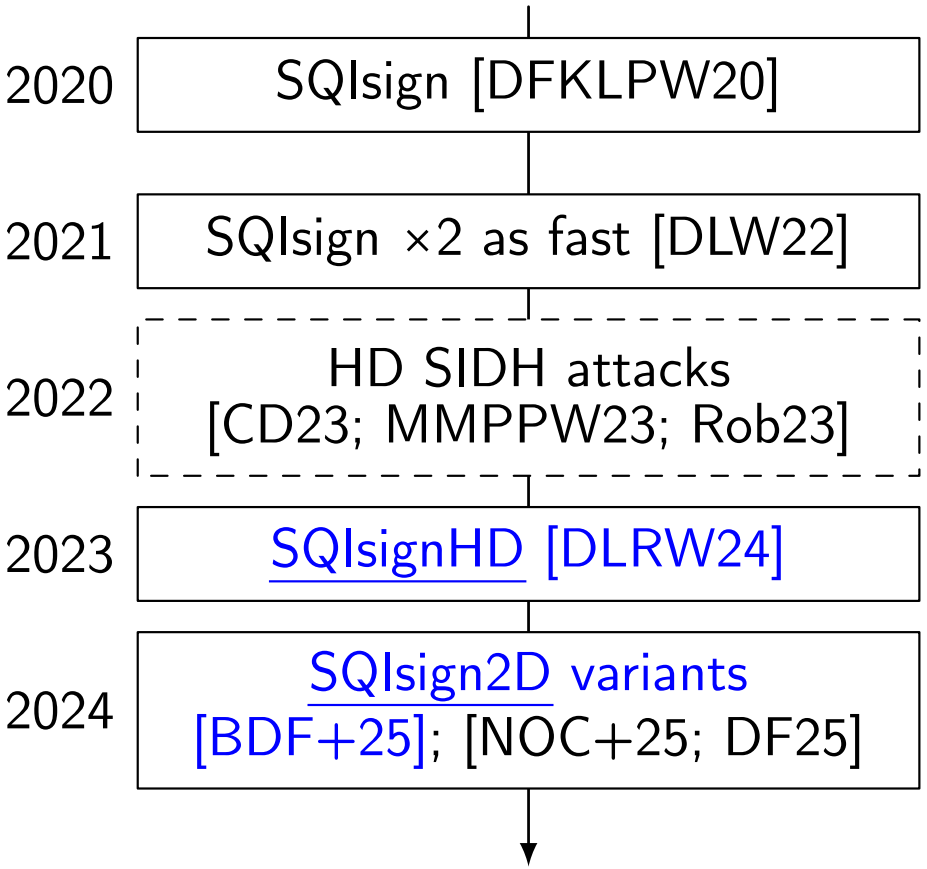


1D Ideal to Isogeny

Enhanced 1D Ideal to Isogeny

Introduction of HD isogenies in cryptology

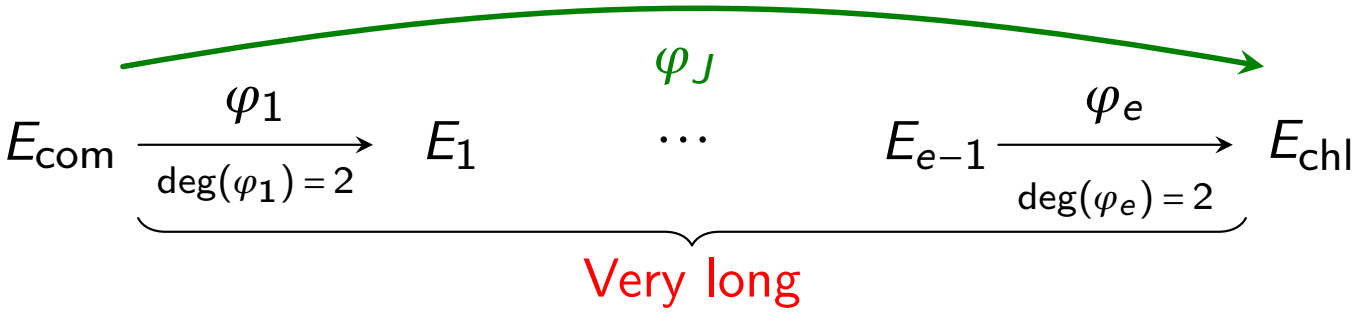
4D Ideal to Isogeny (verif)

2D Ideal to Isogeny (sign & verif)

In blue: our contributions

Ideal to isogeny translation: the HD breakthrough

- **In SQIsign v1.0:** $\mathrm{nrd}(J) = 2^e$ very big



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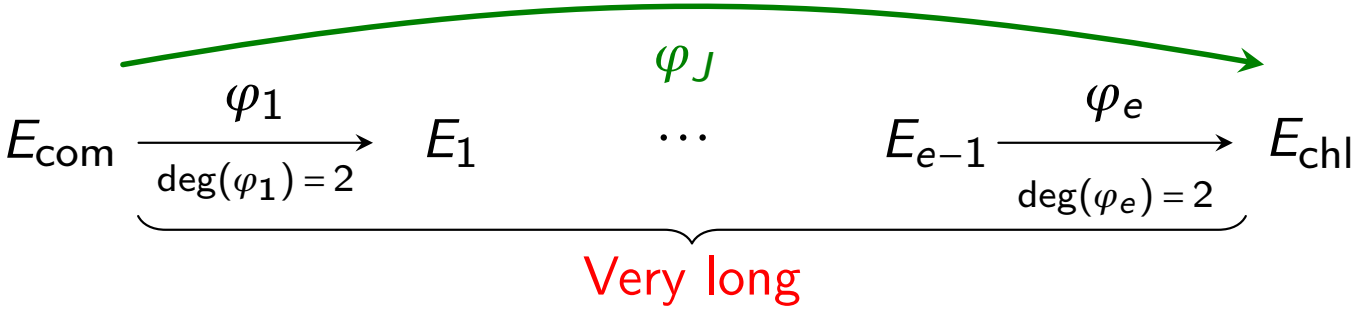

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- **In SQIsignHD/2D:** $\mathrm{nrd}(J)$ small (but not smooth)

Response

P, Q

$\varphi_J(P, Q)$

✓ Fast

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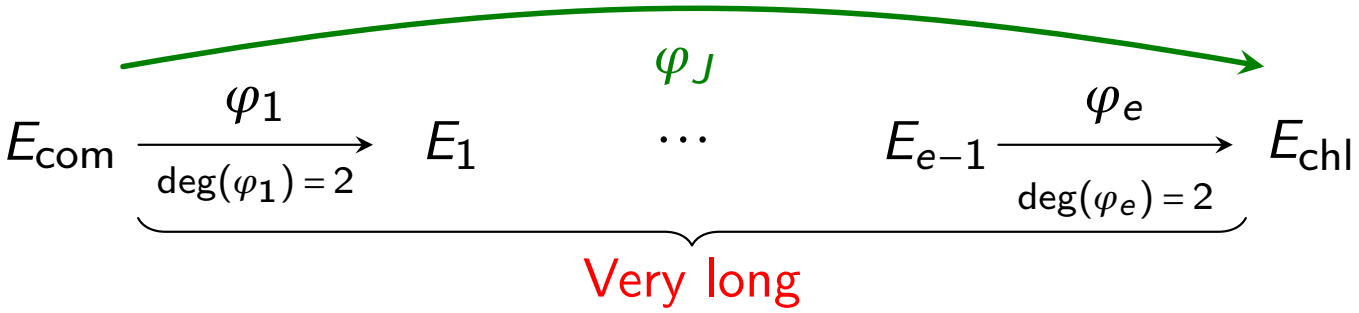

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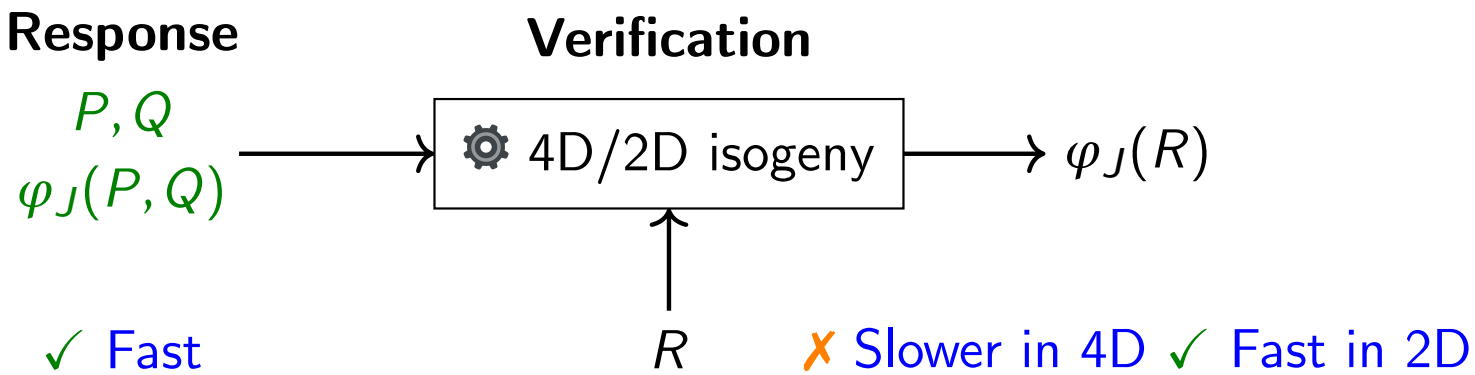
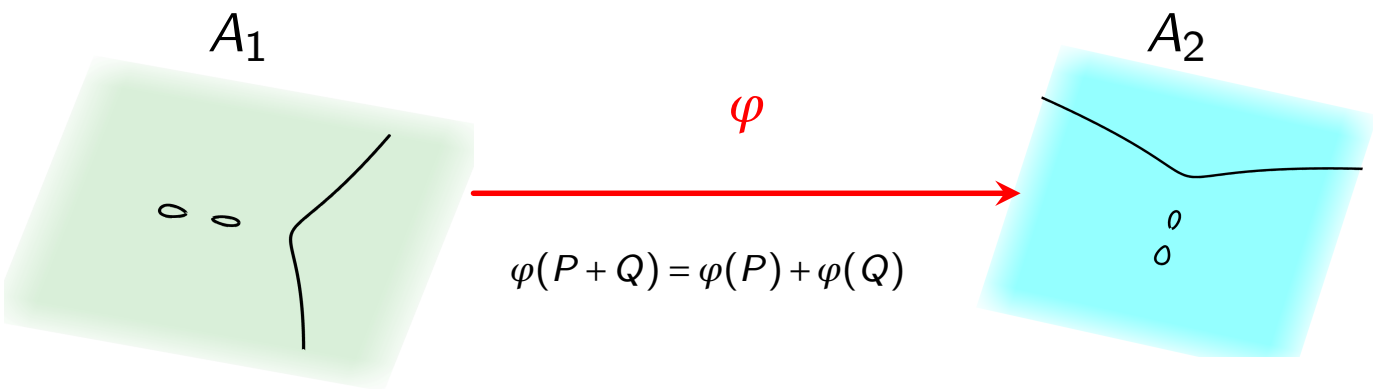


New dimensions in cryptography

Isogenies between abelian varieties

- Abelian varieties are projective abelian group varieties, generalizing elliptic curves.
  Example: $E_1 \times E_2$ is an abelian variety of dimension 2.
- Between abelian varieties, isogenies are morphisms which are surjective and of finite kernel.



An isogeny between abelian surfaces

n-isogenies in higher dimension and their degree

- Let $\varphi : A \longrightarrow B$ be an isogeny between principally polarised abelian varieties (PPAVs).
- Then there exists a *contragradient isogeny* $\widetilde{\varphi} : B \longrightarrow A$ with $\deg(\varphi) = \deg(\widetilde{\varphi})$.

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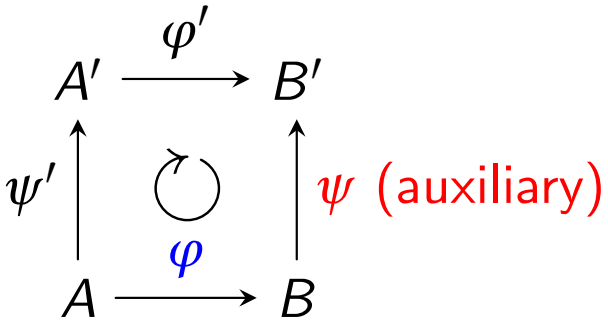
- ⚠ If φ is an n-isogeny, $\deg(\varphi) = n^{\dim(A)} = \deg_p(\varphi)^{\dim(A)}$.
- ⚠ Not all isogenies are n-isogenies ($\deg_p(\varphi)$ is not always defined) in dimension > 1.

Kani's lemma [Kan97]

- Consider the commutative diagram:



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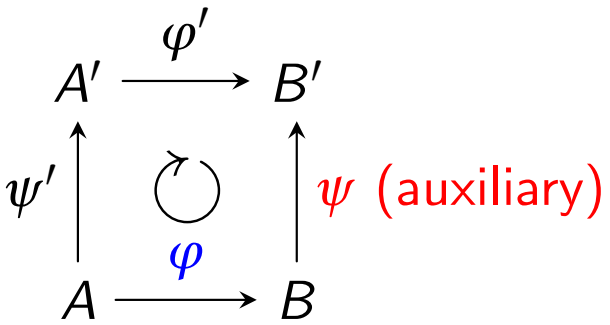

  with $\deg_p(\varphi) = \deg_p(\varphi') = q$ and $\deg_p(\psi) = \deg_p(\psi') = r$.
- Assume that $\gcd(q, r) = 1$ and $q + r = 2^e$.
- Then

$$F := \begin{pmatrix} \varphi & \widetilde{\psi} \\ -\psi' & \widetilde{\varphi'} \end{pmatrix} : A \times B' \longrightarrow B \times A'$$

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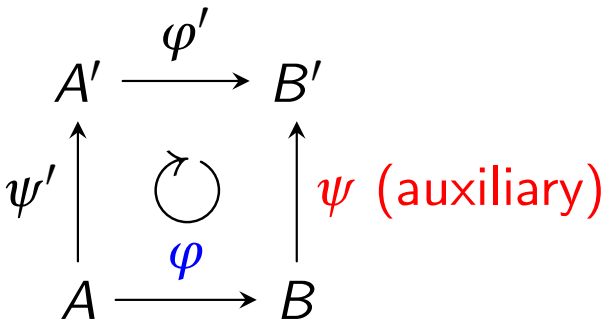

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- Its kernel is:

$$\ker(F) = \{([q]P, \psi \circ \varphi(P)) \mid P \in A[2^e]\},$$

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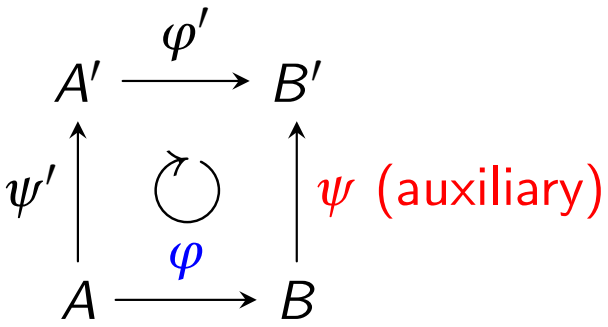

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  so F can be computed from q and $\psi \circ \varphi(A[2^e])$.
- F **efficiently represents** φ:

$$F(P, 0) = (\varphi(P), -\psi'(P)).$$

SQIsignHD - verifying in dimension 4

Goal: Compute $\varphi_{\mathsf{rsp}} : E_{\mathsf{com}} \to E_{\mathsf{chl}}$.

Input: Ideal I connecting $\mathcal{O}_1 \simeq \mathsf{End}(E_{\mathsf{com}})$ and $\mathcal{O}_2 \simeq \mathsf{End}(E_{\mathsf{chl}})$.

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- Solve $\mathsf{nrd}(J) + a_1^2 + a_2^2 = 2^e$ for $J \sim I$ and $a_1, a_2 \in \mathbb{Z}$.
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The verification: Apply Kani's lemma to:

- $\Phi := \mathrm{Diag}(\varphi_J, \varphi_J)$ with $\deg_p(\Phi) = \mathrm{nrd}(J)$.
- The auxiliary isogeny:

$$\Psi := \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} : E_{\mathsf{chl}}^2 \to E_{\mathsf{chl}}^2,$$

with $\deg_p(\Psi) = a_1^2 + a_2^2$.

$$\begin{array}{ccc} E_{\mathsf{com}}^2 & \xrightarrow{\Phi'} & E_{\mathsf{chl}}^2 \\ {\scriptstyle\Psi'}\uparrow & \circlearrowright & \uparrow{\scriptstyle\Psi} \\ E_{\mathsf{com}}^2 & \xrightarrow{\Phi} & E_{\mathsf{chl}}^2 \end{array}$$

SQIsignHD - verifying in dimension 4

The verificarion:

- By Kani's lemma,

$$F := \begin{pmatrix} \Phi & \widetilde{\Psi} \\ -\Psi' & \widetilde{\Phi'} \end{pmatrix} := \begin{pmatrix} a_1 & -a_2 & \widehat{\varphi}_J & 0 \\ a_2 & a_1 & 0 & \widehat{\varphi}_J \\ -\varphi_J & 0 & a_1 & a_2 \\ 0 & -\varphi_J & -a_2 & a_1 \end{pmatrix} : E_{\text{com}}^2 \times E_{\text{chl}}^2 \to E_{\text{com}}^2 \times E_{\text{chl}}^2$$

satisfies $\deg_p(F) = \text{nrd}(J) + a_1^2 + a_2^2 = 2^e$.

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  satisfies $\deg_p(F) = \text{nrd}(J) + a_1^2 + a_2^2 = 2^e$.
- $\ker(F)$ can be computed with input data $a_1, a_2, P, Q, \varphi_J(P), \varphi_J(Q)$.
- Since $\deg_p(F) = 2^e$, to compute F, we decompose

$$E_{\text{com}}^2 \times E_{\text{chl}}^2 \xrightarrow{F_1} A_1 \quad \overset{F}{\cdots} \quad A_{e-1} \xrightarrow{F_e} E_{\text{com}}^2 \times E_{\text{chl}}^2$$

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- To check F represents φ_J correctly, evaluate:

$$F(R,0,0,0) = ([a_1]R, -[a_2]R, -\varphi_J(R), 0)$$

SQIsign2D - verifying in dimension 2

SQIsignHD trade-off:

✓ Very fast signing/response.

✗ Slower verification (4D).

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The solution: SQIsign2D(-West)

- **Issue:** we need an auxiliary isogeny $\psi : E_{\mathsf{chl}} \to E_{\mathsf{aux}}$ of degree $2^e - \mathrm{nrd}(J)$.

$$F := \begin{pmatrix} \varphi_J & \widetilde{\psi} \\ -\psi' & \widetilde{\varphi}' \end{pmatrix} : E_{\mathsf{com}} \times E_{\mathsf{aux}} \to E_{\mathsf{chl}} \times E'$$

- $\ker(F)$ depends on $\mathrm{nrd}(J), P, Q, \psi \circ \varphi_J(P), \psi \circ \varphi_J(Q)$, where (P, Q) is a basis of $E_{\mathsf{com}}[2^e] \simeq (\mathbb{Z}/2^e\mathbb{Z})^2$.
- **Response:** $\psi \circ \varphi_J(P), \psi \circ \varphi_J(Q)$ trickier to compute (2D computation).

The HD trade-off

SQIsign v1.0

Response of smooth and very big degree $\deg(\varphi_{\mathsf{rsp}}) = 2^e$.

VS

SQIsignHD/2D

Response of non-smooth smaller degree $\deg(\varphi_{\mathsf{rsp}})$.

component of

Smooth and small polarised degree HD isogeny.

SQIsign is very compact

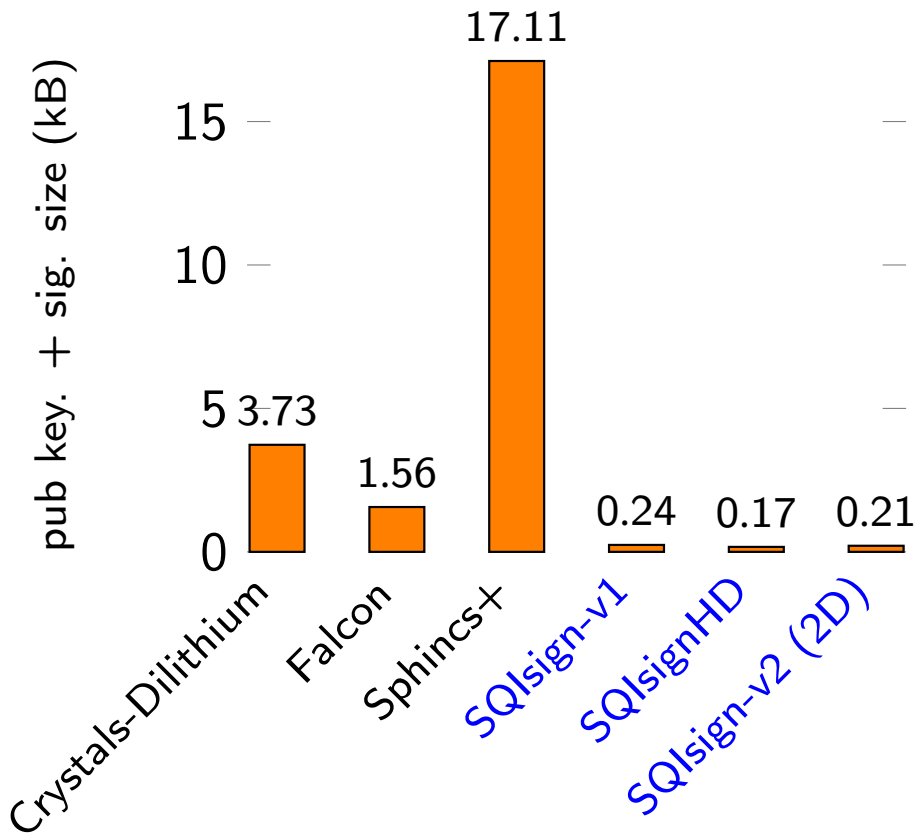

Figure: Compacity of SQIsign and other post-quantum digital signatures standardised by NIST.

In black, standardised post-quantum schemes:

- Crystals-Dilithium (lattices)
- Falcon (lattices)
- Sphincs+ (hash)

In blue, variants of SQIsign.

SQIsign is getting faster

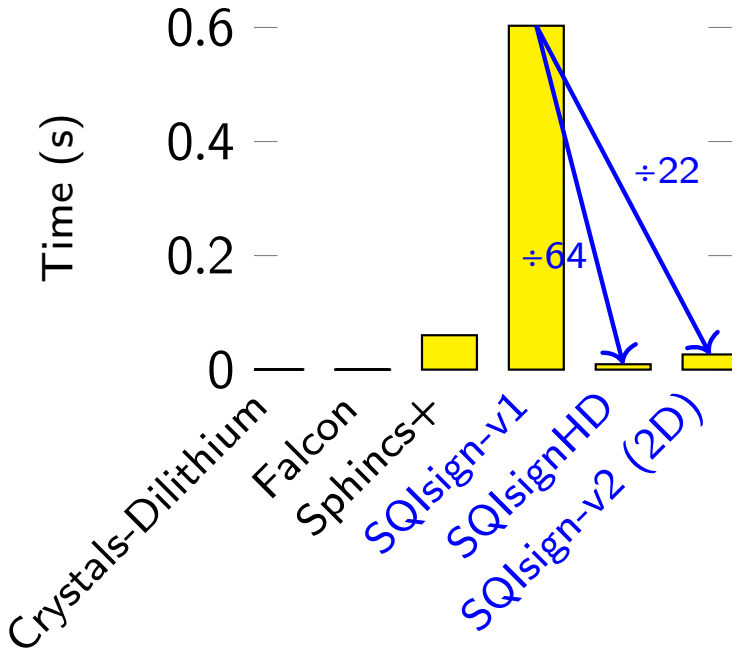

Figure: **Signing** time of SQIsign and post-quantum digital signatures standardised by NIST an a 2.3 GHz CPU.

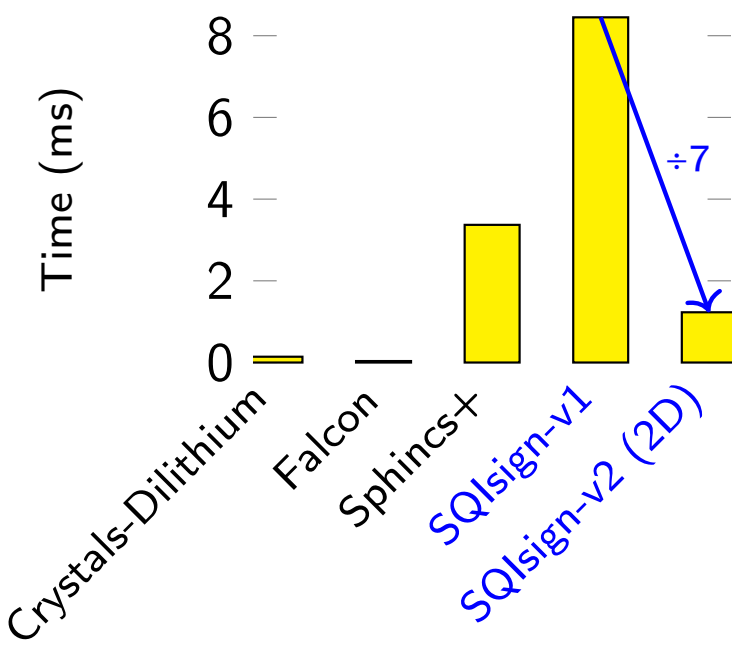

Figure: **Verification** time of SQIsign and post-quantum digital signatures standardised by NIST an a 2.3 GHz CPU.

Computing higher dimensional isogenies

Relies on Mumford's theta coordinates theory (1966).

Contributions:

- **In 2D:** Pierrick Dartois, Luciano Maino, Giacomo Pope and Damien Robert, "An Algorithmic Approach to (2,2)-isogenies in the Theta Model and Applications to Isogeny-based Cryptography", In **ASIACRYPT 2024**.
- **In 4D:** Pierrick Dartois, "Fast computation of 2-isogenies in dimension 4 and cryptographic applications", In **Journal of Algebra**, Volume 683, December 2025.
- **In 4D:** Pierrick Dartois and Max Duparc. "Chasing Rabbits Through Hypercubes: Better algorithms for higher dimensional 2-isogeny computations." Accepted at ANTS 2026.

In blue, contributions of this PhD.

From quaternionic to quadratic ideal action

Idea: Let E be a supersingular elliptic curve.

- Consider a 2-dimensional subring $\mathfrak{O} \hookrightarrow \mathrm{End}(E)$.
- A quadratic ideal $\mathfrak{a} \subseteq \mathfrak{O}$ corresponds to an isogeny $\varphi_{\mathfrak{a}} : E \to E_{\mathfrak{a}}$.

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Solution: use 4-dimensional isogenies

- Pierrick Dartois, Jonathan Komada Eriksen, Tako Boris Fouotsa, Arthur Herlédan Le Merdy, Riccardo Invernizzi, Damien Robert, Ryan Rueger, Frederik Vercauteren and Benjamin Wesolowski, "PEGASIS: Practical Effective Class Group Action using 4-Dimensional Isogenies", In **CRYPTO 2025**.
- Pierrick Dartois, Jonathan Komada Eriksen, Riccardo Invernizzi and Frederik Vercauteren. "qt-Pegasis: Simpler and Faster Effective Class Group Actions." In **EUROCRYPT 2026**.

The Module Isogeny Key Exchange (MIKE)

- Based on module action (generalising ideal action) on abelian varieties.
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Eurocrypt 2026 rump session, May 12, 2026.

Thank you for listening

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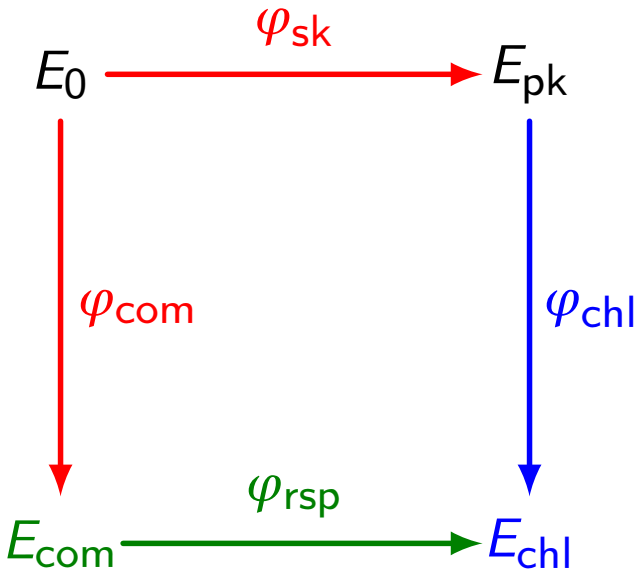
Future works:

- Focus on side channel proof implementation of SQIsign (NIST recommendation).
- Improve HD algorithms (*e.g.* Pegasis in 2D).

You can find my PhD manuscript here:

Response computation in SQIsignHD



Goal: Translate J into $\varphi_{rsp} = \varphi_J$.

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Prover's secret
published by Verifier
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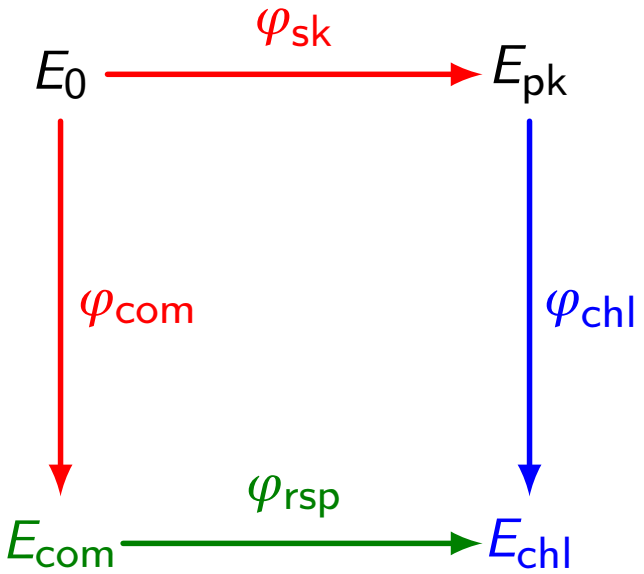
Response computation in SQIsignHD



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- Then $J := I\overline{\alpha}/\mathrm{nrd}(I)$ for some $\alpha \in I$.
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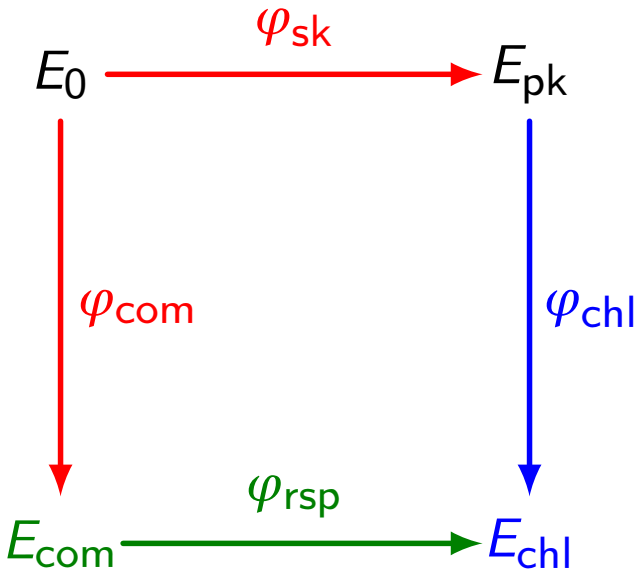
Response computation in SQIsignHD



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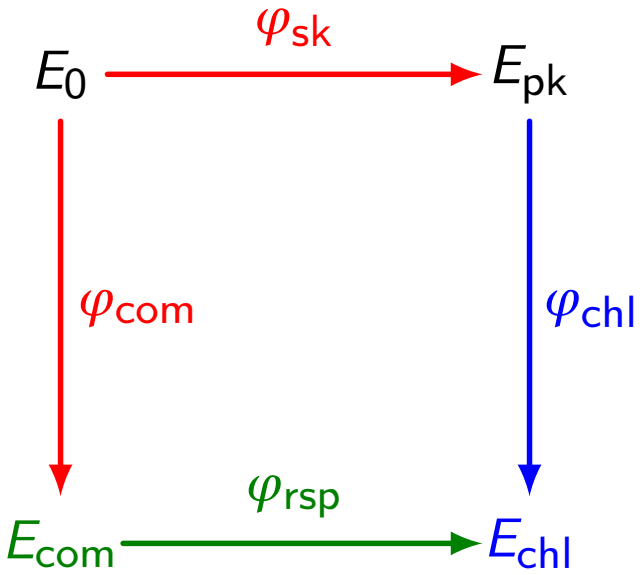
- Write $\deg(\varphi_{chl}) = 2^\lambda$.
- Let (P', Q') be a basis of $E_{com}[2^{e+\lambda}]$ and let $(P, Q) := [2^\lambda](P', Q')$.

Response computation in SQIsignHD



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- Then

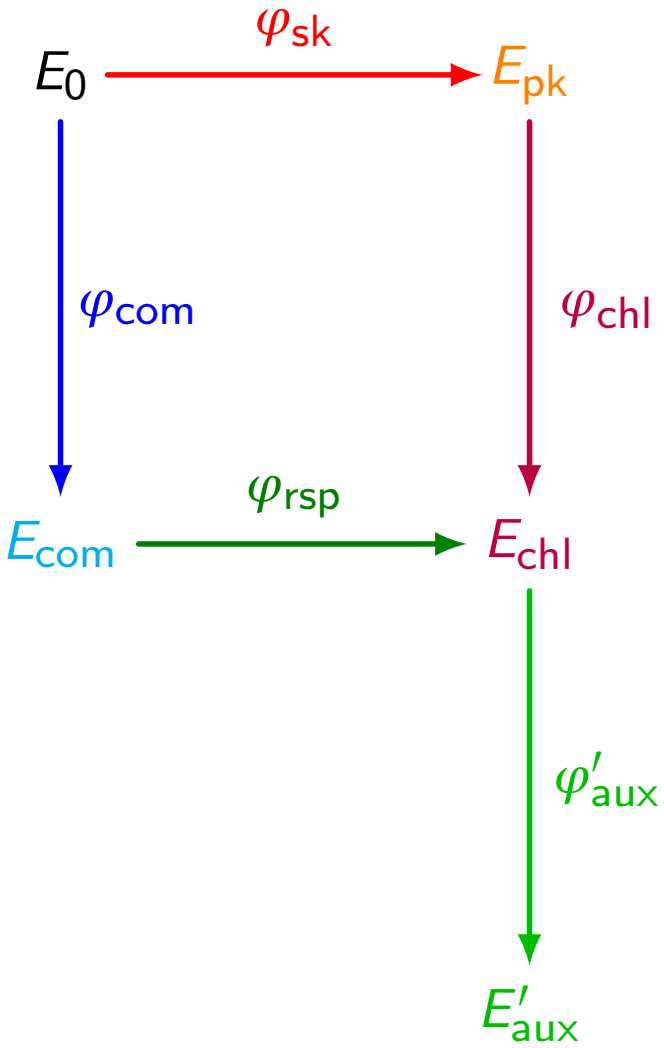
$$\varphi_{rsp}(P, Q) = [\mu]\varphi_{chl} \circ \varphi_{sk} \circ \widehat{\alpha} \circ \widehat{\varphi}_{com}(P', Q'),$$

with $\mu \equiv (\deg(\varphi_{com})\deg(\varphi_{sk}))^{-1} \bmod 2^e$.

Improvements of SQIsign security assumptions

	SQIsign	SQIsignHD	SQIsign2D
Soundness	The Endomorphism Ring Problem (strong)		
Zero knowledge	• Heuristic on the distribution of φ_{rsp}.	• An oracle returning "random" isogenies. • Heuristic on the distribution of E_{com} (uniform).	• 2 oracles returning "random" isogenies.

Response/signature



Response:

- Compute $I_{chl} \subset \mathrm{End}(E_{pk})$ associated to φ_{chl}.
- $J \longleftarrow \overline{I}_{com} \cdot I_{sk} \cdot I_{chl}$.
- Compute $I_{rsp} \sim J$ random of norm $q < 2^r \simeq \sqrt{p}$.
- Sample $I'_{aux} \subset \mathrm{End}(E_{chl})$ at random of norm $2^r - q$.
- Translate $I_{com} \cdot I_{rsp} \cdot I'_{aux}$ into $\varphi'_{aux} \circ \varphi_{rsp} \circ \varphi_{com}$.

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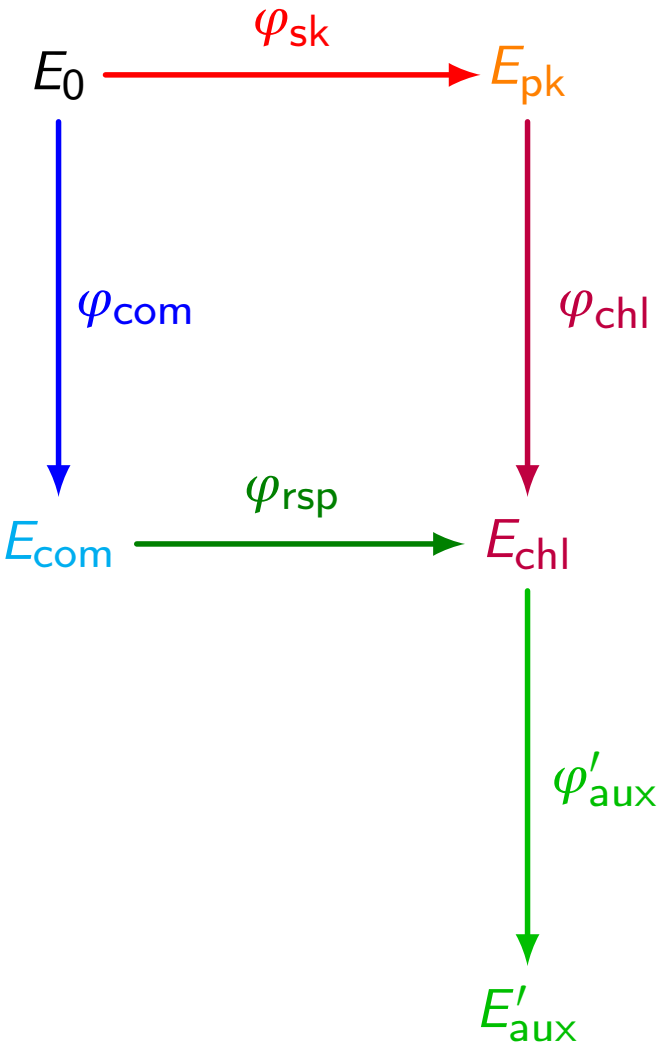

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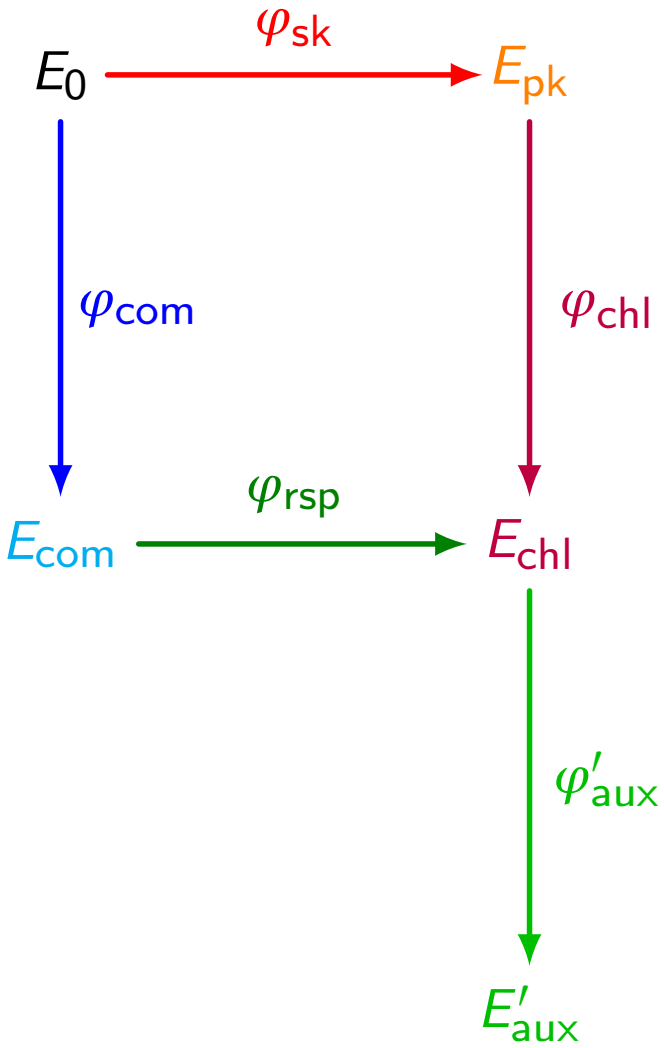
Response/signature



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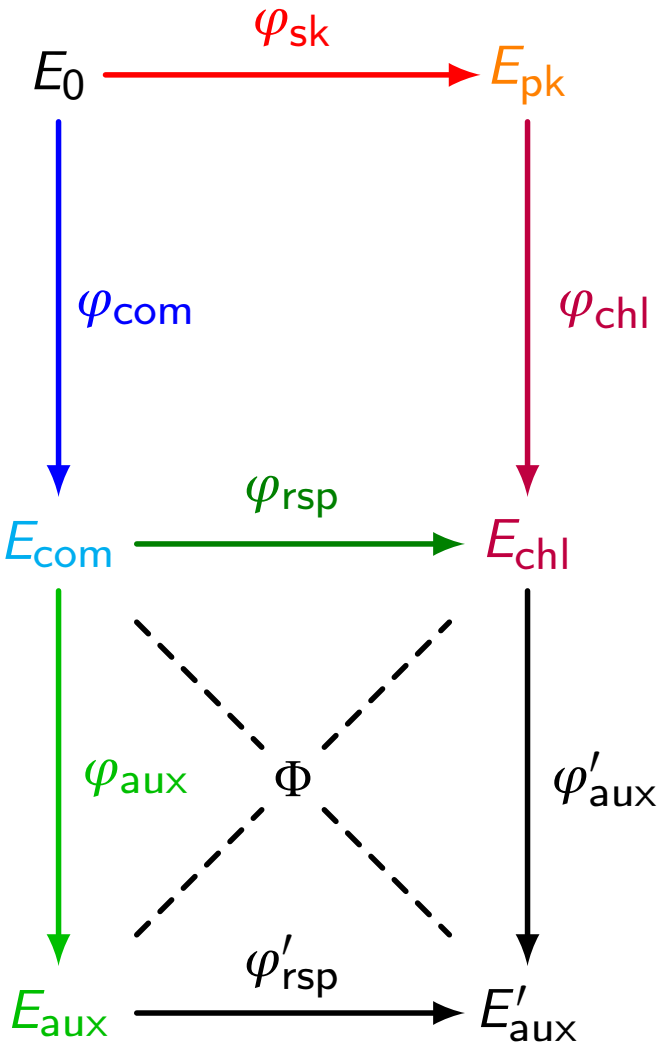
Signature: Could be

$$(E_{com}, E'_{aux}, \varphi'_{aux} \circ \varphi_{rsp}(E_{com}[2^r])).$$

Response/signature - commitment recoverability



Response/signature:

- Compute the $(2^r, 2^r)$-isogeny:

$$\Phi : E_{com} \times E'_{aux} \longrightarrow E_{chl} \times E_{aux}$$

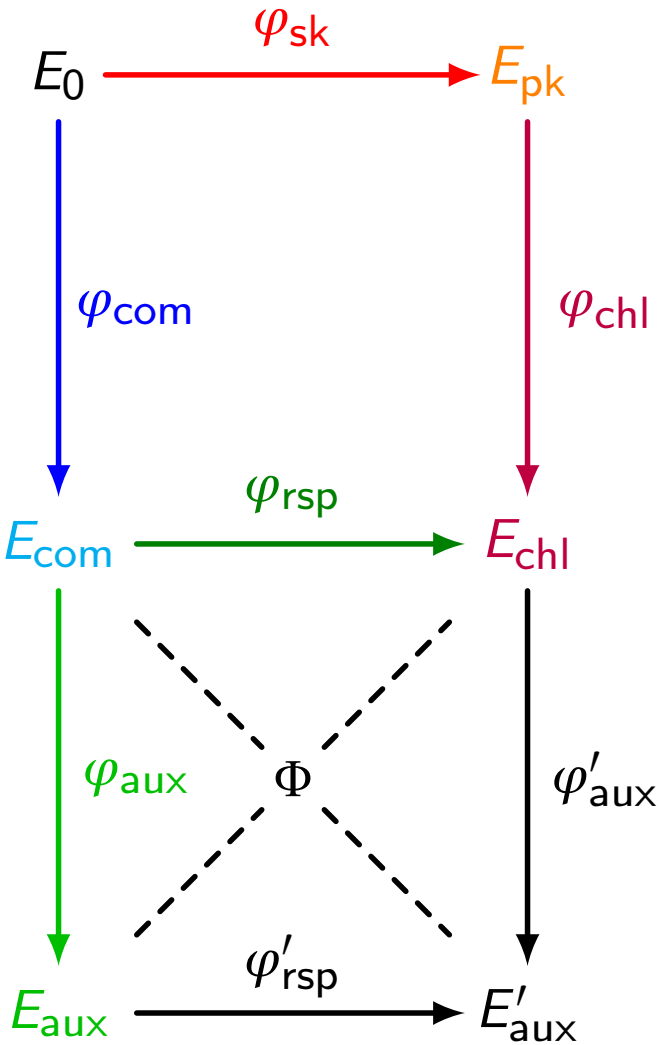
from $\varphi'_{aux} \circ \varphi_{rsp}(E_{com}[2^r])$.

- Evaluate Φ to compute $\varphi_{aux} \circ \widehat{\varphi}_{rsp}(E_{chl}[2^r])$.

Signature:

$$(E_{aux}, \varphi_{aux} \circ \widehat{\varphi}_{rsp}(E_{chl}[2^r])).$$

Verification



Verification:

- Compute the $(2^r, 2^r)$-isogeny:

  $$\widehat{\Phi} : E_{chl} \times E_{aux} \longrightarrow E_{com} \times E'_{aux}$$

  from $\varphi_{aux} \circ \widehat{\varphi}_{rsp}(E_{chl}[2^r])$.
- Check its codomain is $E_{com} \times \_$.