Fast computation of higher dimensional isogenies for cryptographic applications

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PhD Defense
Under the supervision of Damien Robert, Benjamin Wesolowski and Luca De Feo

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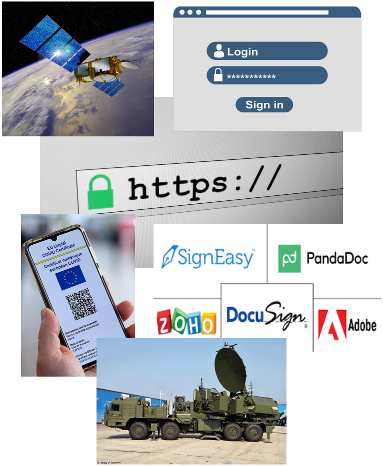
Cryptography in daily life

What people think crypto is:

What crypto really is:



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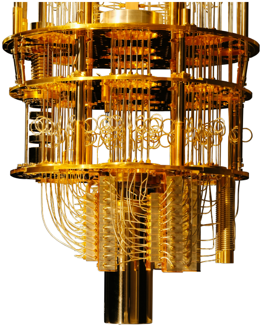
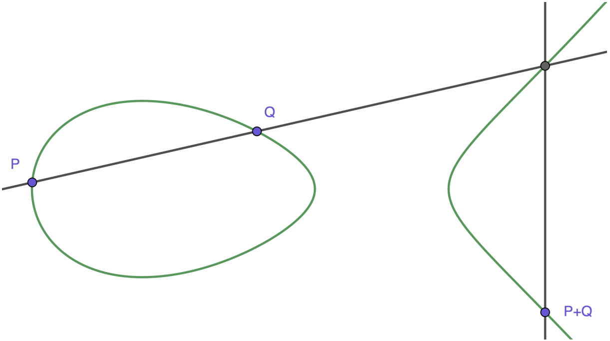
...All broken by quantum computers



Elliptic curves

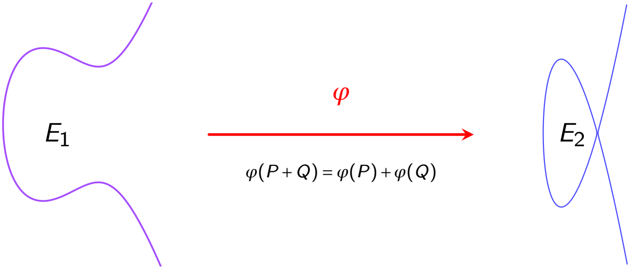


- An elliptic curve $E/\mathbb{F}_q$ is defined by:

$$y^2 = x^3 + ax + b, \quad a, b \in \mathbb{F}_q$$

  with an infinite element 0_E.
- E is equipped with a commutative group law.

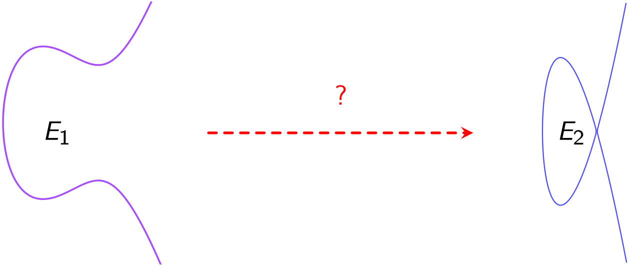
Isogenies



$$\varphi(x,y)=\left(\frac{p(x)}{q(x)},y\frac{r(x)}{s(x)}\right)$$

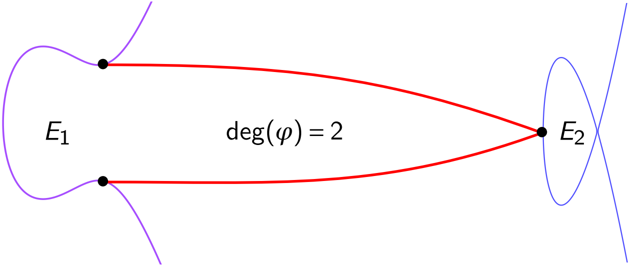
Why are isogenies interesting in cryptography?

The isogeny problem: Given two elliptic curves $E_1, E_2/\mathbb{F}_q$, find an isogeny $E_1 \longrightarrow E_2$.



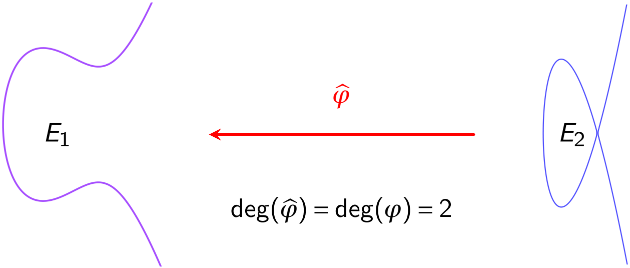
This problem is assumed to be hard for both classical and quantum computers.

Isogenies - degree



An isogeny of degree n is called an *n-isogeny*.

Isogenies - the dual isogeny



$\deg(\widehat{\varphi}) = \deg(\varphi) = 2$

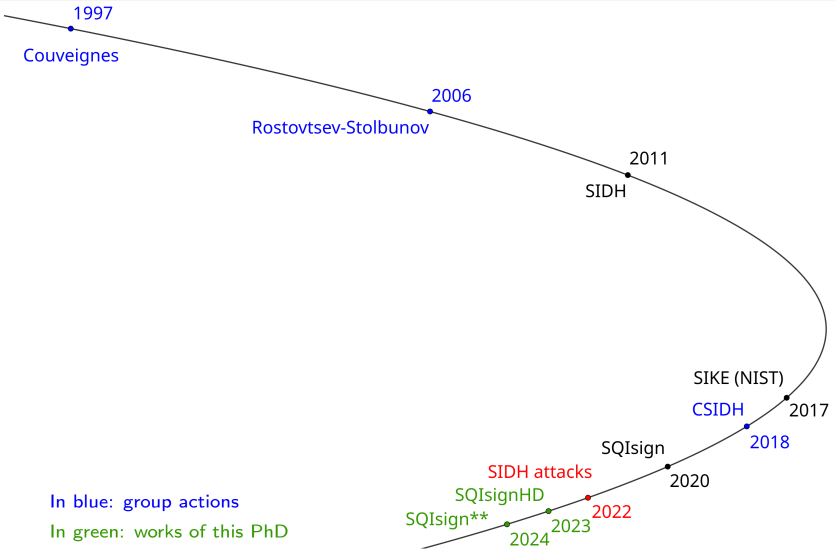
An n-isogeny φ satisfies $\widehat{\varphi} \circ \varphi = [n]$.

Isogeny chains

$$E_0 \xrightarrow{\varphi_1} E_1 \qquad \cdots \qquad E_{n-1} \xrightarrow{\varphi_n} E_n$$

$$\deg(\varphi_n \circ \cdots \circ \varphi_1) = \prod_{i=1}^{n} \deg(\varphi_i)$$

A brief (biased) history of isogeny based cryptography



In blue: group actions

In green: works of this PhD

SQIsign and the Deuring correspondence

The Endomorphism ring

Definition (Endomorphism ring)

$$\mathrm{End}(E) = \{0\} \cup \{\text{Isogenies } \varphi : E \longrightarrow E\}$$

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Theorem (Deuring)

Let $E/\mathbb{F}_q$ ($p = \mathrm{char}(\mathbb{F}_q)$). Then $\mathrm{End}(E)$ is either isomorphic to:

- *An order in a quadratic imaginary field. We say that E is ordinary.*
- *A maximal order in a quaternion algebra ramifying at p and ∞. We say that E is supersingular.*

Quaternions - Definitions

- **Quaternion algebra ramifying at p and ∞:** A 4-dimensional non commutative division algebra over $\mathbb{Q}$:

$$\mathscr{B}_{p,\infty} = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}k,$$

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$$i^2 = -1 \text{ (if } p \equiv 3 \mod 4), \quad j^2 = -p \quad \text{and} \quad k = ij = -ji.$$

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- **Order:** A full rank lattice $\mathscr{O} \subset \mathscr{B}_{p,\infty}$ with a ring structure.
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- **Left Ideal:** A left $\mathcal{O}$-ideal I is a full rank lattice $I \subset \mathcal{B}_{p,\infty}$ such that $\mathcal{O} \cdot I = I$.
- **Right Ideal:** A right $\mathcal{O}$-ideal I is a full rank lattice $I \subset \mathcal{B}_{p,\infty}$ such that $I \cdot \mathcal{O} = I$.

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- **Equivalent left $\mathcal{O}$-ideals:** $I \sim J \Longleftrightarrow \exists \alpha \in \mathcal{B}_{p,\infty}^*, \quad J = I\alpha$.

The Deuring correspondence

Supersingular elliptic curves	**Quaternions**
$j(E)$ or $j(E)^p$ supersingular	$\mathscr{O} \cong \mathrm{End}(E)$ maximal order in $\mathscr{B}_{p,\infty}$

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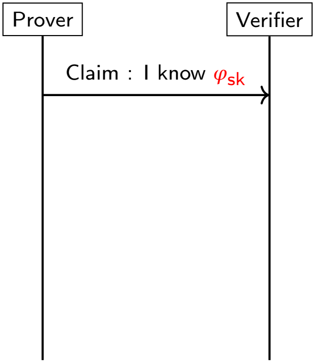
$\deg(\varphi)$	$\mathrm{nrd}(I_\varphi)$

The SQIsign identification scheme

$$E_0 \xrightarrow{\varphi_{\mathrm{sk}}} E_{\mathrm{pk}}$$



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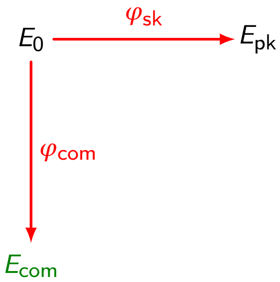

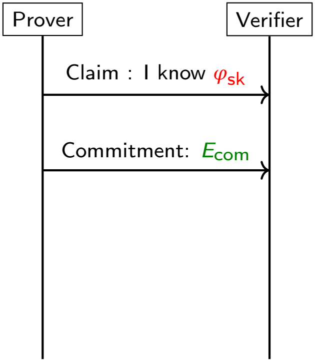

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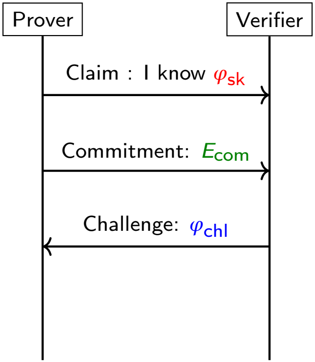

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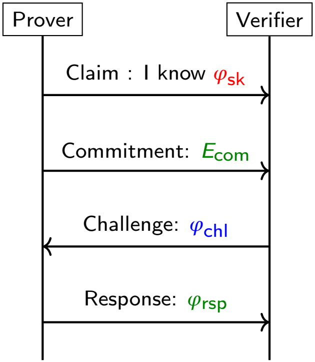

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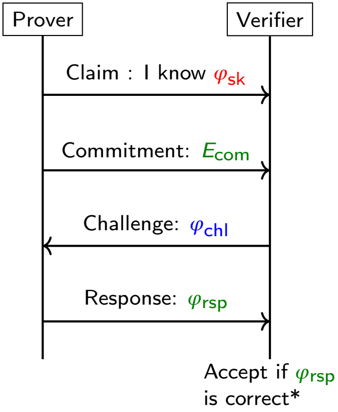

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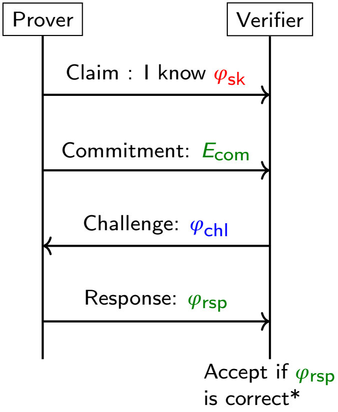

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Computing isogenies via the Deuring correspondence

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Problem: How to compute isogenies between elliptic curves of known endomorphism rings?

- Let E_1 and E_2 of known endomorphism rings $\mathcal{O}_1 \cong \mathrm{End}(E_1)$ and $\mathcal{O}_2 \cong \mathrm{End}(E_2)$.
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✗ Slow in practice because of the red steps.

What does it mean to "compute" an isogeny?

Definition (Efficient representation)

Let $\varphi : E \longrightarrow E'$ be a d-isogeny over $\mathbb{F}_q$. An efficient representation of φ with respect to an algorithm $\mathscr{A}$ is some data $D_\varphi \in \{0,1\}^*$ such that:

1. D_φ has size $\mathrm{poly}(\log(d), \log(q))$.
2. For all $P \in E(\mathbb{F}_{q^k})$, $\mathscr{A}(D_\varphi, P)$ returns $\varphi(P)$ in time $\mathrm{poly}(\log(d), k\log(q))$.

What does it mean to "compute" an isogeny?

Examples of efficient representations:

- If $\deg(\varphi) = \prod_{i=1}^{r} \ell_i$, a chain of isogenies:

$$\begin{tikzcd} E_0 \arrow[r, "\varphi_1", "\deg(\varphi_1)=\ell_1"'] \arrow[rrr, bend left, "\varphi"] & E_1 & \cdots & E_{n-1} \arrow[r, "\varphi_n", "\deg(\varphi_n)=\ell_n"'] & E_n \end{tikzcd}$$

- If $\deg(\varphi)$ is smooth, a generator $P \in E(\mathbb{F}_q)$ s.t. $\ker(\varphi) = \langle P \rangle$ (Vélu).

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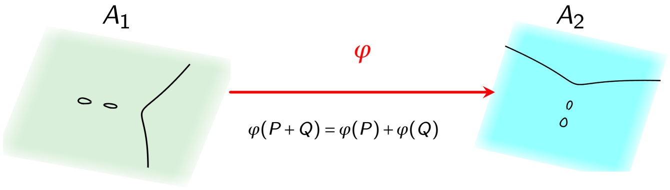
- **New:** If $\deg(\varphi) < 2^e$ is odd and $E[2^e] = \langle P, Q \rangle$, the image points $(\varphi(P), \varphi(Q))$ (higher dimensional interpolation).

New dimensions in cryptography

Isogenies between abelian varieties

- Abelian varieties are projective abelian group varieties, generalizing elliptic curves.
- Between abelian varieties, isogenies are morphisms which are surjective and of finite kernel.



An isogeny between abelian surfaces

n-isogenies in higher dimension

- Let $\varphi : A \longrightarrow B$ be an isogeny between principally polarised abelian varieties (PPAVs).
- Then there exists a *contravariant isogeny* $\widetilde{\varphi} : B \longrightarrow A$ with $\deg(\varphi) = \deg(\widetilde{\varphi})$.

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- φ is an n-isogeny if $\widetilde{\varphi} \circ \varphi = [n]$.
- ⚠ This is not a general fact.
- ⚠ n-isogenies have degree n^g (with $g = \dim(A) = \dim(B)$).

Kani's lemma (dimension 2) [Kan97]

Consider the following commutative diagram:

$$\begin{tikzcd} E_4 \arrow[r, "\varphi'"] & E_3 \\ E_1 \arrow[u, "\psi'"] \arrow[r, "\varphi"'] \arrow[ur, phantom, "\circlearrowright"] & E_2 \arrow[u, "\psi"'] \end{tikzcd}$$

s.t. $\deg(\varphi) = \deg(\varphi') = q$ and $\deg(\psi) = \deg(\psi') = r$ are coprime.

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s.t. $\deg(\varphi) = \deg(\varphi') = q$ and $\deg(\psi) = \deg(\psi') = r$ are coprime. Then the isogeny:

$$\Phi := \begin{pmatrix} \varphi & \widehat{\psi} \\ -\psi' & \widehat{\varphi}' \end{pmatrix} : E_1 \times E_3 \longrightarrow E_2 \times E_4$$

is a $(q+r)$-isogeny, i.e. $\widetilde{\Phi} \circ \Phi = [q+r]$, and its kernel is:

$$\ker(\Phi) = \{([q]P, \psi \circ \varphi(P)) \mid P \in E_1[q+r]\}.$$

Kani's lemma (dimension 2) [Kan97]

- Let $\varphi : E_1 \longrightarrow E_2$ be an isogeny of odd degree $q < 2^e$ to be computed.
- Let $\psi : E_2 \longrightarrow E_3$ be an auxiliary isogeny of degree $r := 2^e - q$.

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- So we can compute

$$\Phi := \begin{pmatrix} \varphi & \widehat{\psi} \\ -\psi' & \widehat{\varphi}' \end{pmatrix} : E_1 \times E_3 \longrightarrow E_2 \times E_4$$

as a chain of e 2-isogenies [DMPR25]:

$$E_1 \times E_3 \xrightarrow{\Phi_1} A_1 \xrightarrow{\Phi_2} A_2 \quad \cdots \quad A_{e-1} \xrightarrow{\Phi_e} E_2 \times E_4.$$

Kani's lemma [Kan97] and efficient representations

- Knowing Φ, we can evaluate φ everywhere:

$$\Phi(P,0) = (\varphi(P), -\psi'(P)).$$

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The Power of Kani's lemma:

- A way to interpolate isogenies given their images on torsion points (led to SIDH attacks).
- Provides efficient representations on non-smooth degree isogenies.

HD techniques for the Deuring correspondence

Problem: How to compute isogenies between elliptic curves of known endomorphism rings?

- Let E_1 and E_2 of known endomorphism rings $\mathcal{O}_1 \cong \mathrm{End}(E_1)$ and $\mathcal{O}_2 \cong \mathrm{End}(E_2)$.
- Compute a connecting ideal I between $\mathcal{O}_1$ and $\mathcal{O}_2$ (left $\mathcal{O}_1$-ideal and right $\mathcal{O}_2$-ideal).
- Compute $J \sim I$ random of ~~smooth norm via [KLPT14]~~ of (small) norm.
- Translate J into an isogeny $\varphi_J : E_1 \longrightarrow E_2$ using dimension 2 or 4 interpolation techniques.

✓ Takes polynomial time.

✓ Becomes hard when $\mathrm{End}(E_1)$ or $\mathrm{End}(E_2)$ is unknown.

✓ Faster than the previous method.

The Clapoti method [PR23]

Goal: Given $E_0/\mathbb{F}_{p^2}$ of equation $y^2 = x^3 + x$ and known endomorphism ring $\mathscr{O}_0$, and a left $\mathscr{O}_0$-ideal I, compute $\varphi_I : E_0 \longrightarrow E_I$.

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$\mathfrak{O}$-ideals	$\mathfrak{O}$-oriented curves and isogenies
Ideal $\mathfrak{a} \subseteq \mathfrak{O}$	$\varphi_{\mathfrak{a}} : E \longrightarrow E_{\mathfrak{a}} := \mathfrak{a} \cdot E$
$\mathfrak{b} \sim \mathfrak{a}$	$\mathfrak{a} \cdot E \simeq \mathfrak{b} \cdot E$
$\alpha\mathfrak{O}$	$\iota(\alpha) : E \longrightarrow E$
$\overline{\mathfrak{a}}$	$\widehat{\varphi}_{\mathfrak{a}}$
$\mathfrak{a}\mathfrak{b}$	$\varphi_{\mathfrak{b}} \circ \varphi_{\mathfrak{a}}$
$N(\mathfrak{a})$	$\deg(\varphi_{\mathfrak{a}})$

Effective group action

Definition

An *effective group action* (EGA) $G \curvearrowright X$ is:

1. Commutative, free and transitive.
2. *Easy to compute*: $g \cdot x$ can be evaluated in polynomial time for all $g \in G$ and $x \in X$.
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- With effective group actions, we can derive many schemes (including key exchange, signatures and more).
- Actually, group actions based on orientations are *restricted* effective group actions. We can act by ideals of small norms $\mathfrak{l}_1, \cdots, \mathfrak{l}_t$ that generate $\mathrm{Cl}(\mathfrak{O})$.
- ⚠ **Issue:** This makes schemes less efficient and less scalable to bigger parameters.

The Clapoti method in PEGASIS [DEF+25]

Goal: Given an $\mathfrak{O}$-oriented curve E and **any** ideal $\mathfrak{a} \subseteq \mathfrak{O}$, compute $E_{\mathfrak{a}} := \mathfrak{a} \cdot E$.

$$\begin{tikzcd}
E'^2 \arrow[rr] & & A_v \\
& & E^2 \arrow[u, "\Phi_v"'] \\
A_u \arrow[uu] \arrow[r, "\widetilde{\Phi}_u"] & E^2 \arrow[r, "\Phi_{\mathfrak{b}}"] & E^2_{\mathfrak{a}} \arrow[u, "\widetilde{\Phi}_{\mathfrak{c}}"']
\end{tikzcd}$$

$$F : A_u \times A_v \longrightarrow E^2_{\mathfrak{a}} \times E'^2$$

- Find $u, v > 0$ and $\mathfrak{b}, \mathfrak{c} \sim \mathfrak{a}$ such that:

$$u\,\mathrm{nrd}(\mathfrak{b}) + v\,\mathrm{nrd}(\mathfrak{c}) = 2^e.$$

- Compute u and v-isogenies Φ_u and Φ_v in dimension 2.
- By Kani's lemma, there exists a 2^e-isogeny $F : A_u \times A_v \longrightarrow E^2_{\mathfrak{a}} \times E'^2$ that embeds $\Phi_{\mathfrak{b}} \circ \widetilde{\Phi}_u$ and $\Phi_{\mathfrak{c}} \circ \widetilde{\Phi}_v$.
- $\ker(F)$ can be computed from Φ_u, Φ_v and $\theta := \widehat{\varphi}_{\mathfrak{c}} \circ \varphi_{\mathfrak{c}}$ that generates $\mathfrak{b} \cdot \overline{\mathfrak{c}}$.
- From F, we extract $E_{\mathfrak{a}}$.

Fast computation of higher dimensional isogenies

Definition: symplectic isomorphism

- Let A/k be a PPAV of dimension g.
- If $n \nmid \mathrm{char}(k)$, then $A[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$.

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- A *symplectic isomorphism* $\varphi : (\mathbb{Z}/n\mathbb{Z})^g \times \widehat{(\mathbb{Z}/n\mathbb{Z})^g} \xrightarrow{\sim} A[n]$ is a group isomorphism satisfying:

$$\forall x, y \in (\mathbb{Z}/n\mathbb{Z})^g \times \widehat{(\mathbb{Z}/n\mathbb{Z})^g}, \quad e_n(\varphi(x), \varphi(y)) = e_n(x, y),$$

where the first pairing is the Weil-pairing and the second one is given by:

$$\forall (i, \chi), (i', \chi') \in (\mathbb{Z}/n\mathbb{Z})^g \times \widehat{(\mathbb{Z}/n\mathbb{Z})^g}, \quad e_n((i, \chi), (i', \chi')) = \chi'(i)\chi(i')^{-1}.$$

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- Such a symplectic isomorphism is determined by a (ζ-)*symplectic basis* $(S_1, \cdots, S_g, T_1, \cdots, T_g)$ of $A[n]$ *i.e.* a basis such that:

$$\forall 1 \leq i, j \leq g, \quad e_n(S_i, S_j) = e_n(T_i, T_j) = 1 \quad \text{and} \quad e_n(S_i, T_j) = \zeta^{\delta_{i,j}},$$

where ζ is a primitive n-th root of unity.

Definition: theta structure

Definition (Max Duparc)

Let A be a PPAV of dimension g. A *(symmetric) theta structure* of level n is a map

$$\begin{array}{rcl} \Theta(n) : A & \longrightarrow & \mathbb{P}^{n^g-1} \\ x & \longmapsto & (\theta_i(x))_{i\in(\mathbb{Z}/n\mathbb{Z})^g} \end{array}$$

along with a symplectic isomorphism:

$$\overline{\Theta}(n) : (\mathbb{Z}/n\mathbb{Z})^g \times \widehat{(\mathbb{Z}/n\mathbb{Z})^g} \xrightarrow{\sim} A[n]$$

satisfying the *theta group action relation*:

$$\theta_i(x+\overline{\Theta}(n)(j,\chi)) = \chi(i)^{-1}\theta_{i+j}(x),$$

for all $x \in A$, $i,j \in (\mathbb{Z}/n\mathbb{Z})^g$ and $\chi \in \widehat{(\mathbb{Z}/n\mathbb{Z})^g}$.

Properties of theta structures

Theta structures are induced by symplectic isomorphisms

Theorem (Mumford, 1966)

A level n theta structure $(\Theta(n), \overline{\Theta}(n))$ on a PPAV A is fully determined by a symplectic isomorphism $\overline{\Theta}(2n) : (\mathbb{Z}/2n\mathbb{Z})^g \times \widehat{(\mathbb{Z}/2n\mathbb{Z})^g} \xrightarrow{\sim} A[2n]$ inducing $\overline{\Theta}(n)$ i.e. by a symplectic basis of $A[2n]$ inducing $\overline{\Theta}(n)$.

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Theta structures and theta null points:

- When $4|n$, the *marked AV* (PPAV and theta structure) $(A, \Theta(n), \overline{\Theta}(n))$ is determined by the *theta null point* $(\theta_i(0_A))_i$.

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- In other cases, we still use the theta null point as a representative of a marked AV.
- This is enough for arithmetic operations.

Theta structures of level 2

Theorem

Let $(A, \Theta(n), \overline{\Theta}(n))$ be a marked AV of level n and dimension g. Then:

1. *[Mum74] If $n \geq 3$, then $\Theta(n) : A \hookrightarrow \mathbb{P}^{n^g-1}$ is an embedding.*
2. *[BL04] If $n = 2$ and A is not a product, then $\Theta(2)$ defines an embedding $A/\pm \hookrightarrow \mathbb{P}^{2^g-1}$.*
3. *[BL04] If $n = 2$ and $A \simeq A_1 \times \cdots \times A_m$, then $\Theta(2)$ defines an embedding*
$$A_1/\pm \times \cdots \times A_m/\pm \hookrightarrow \mathbb{P}^{2^g-1}.$$

Our goal

Goal: Given the kernel $K \subset A[2^e]$ of a 2^e-isogeny between PPAVs $f : A \longrightarrow B$, compute f in level 2 theta coordinates:

$$(\theta_i^A(x))_{i\in(\mathbb{Z}/2\mathbb{Z})^g} \longmapsto (\theta_i^B(f(x)))_{i\in(\mathbb{Z}/2\mathbb{Z})^g}$$

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Method:

- Decompose f as a chain of 2-isogenies:

$$A_0 = A \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \quad \cdots \quad A_{e-1} \xrightarrow{f_e} A_e = B$$

- Compute every 2-isogeny iteratively, using:

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Technicality: We need more torsion $K \subset A[2^{e+2}]$ above the kernel.

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- For that reason, we need 8-torsion points $T_1, \cdots, T_g$ such that $\ker(f) = \langle [4]T_1, \cdots, [4]T_g \rangle$ to compute f.

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- With this data, we compute the codomain theta-null point $(\theta_i(0_B))_i$.

2-isogeny evaluation algorithm

A very simple isogeny evaluation algorithm:

$$(\theta_i^A(x))_i \xrightarrow{H} * \xrightarrow{S} * \xrightarrow{\star(1/\widetilde{\theta}_i^B(0_B))_i} * \xrightarrow{H} (\theta_i^B(f(x)))_i$$

where:

- $H : (x_i)_i \longmapsto \left(\sum_{i\in(\mathbb{Z}/2\mathbb{Z})^g}(-1)^{\langle i|j\rangle}x_i\right)_j$ (Hadamard).
- $S : (x_i)_i \longmapsto (x_i^2)_i$.
- $(x_i)_i \star (y_i)_i := (x_i y_i)_i$.
- $(\widetilde{\theta}_i^B(0_B))_i = H((\theta_i^B(0_B))_i)$ (dual theta null point).

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$$\theta_{i,j}^{A_1 \times A_2}(x, y) = \theta_i^{A_1}(x) \cdot \theta_j^{A_2}(y).$$

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Solution 1: Compute a new theta structure $\Theta'^{A_1 \times A_2}$ such that

$$\overline{\Theta'}^{A_1 \times A_2}(\{0\} \times \widehat{(\mathbb{Z}/2\mathbb{Z})^g}) = \ker(f).$$

Change of coordinate formulas

$$\begin{tikzcd}
\text{ζ-symplectic basis } \mathscr{B} \text{ of } A[4] \arrow[r, "\text{Change of basis}", "\text{matrix } M \in \mathrm{Sp}_{2g}(\mathbb{Z}/4\mathbb{Z})"'] \arrow[d] & \text{ζ-symplectic basis } \mathscr{B}' \text{ of } A[4] \arrow[d] \\
\text{Theta structure } (\Theta^A, \overline{\Theta}^A) \arrow[d] & \text{Theta structure } (\Theta'^A, \overline{\Theta'}^A) \arrow[d] \\
\text{Initial theta coordinates } (\theta_i^A)_i \arrow[r, "\text{Change of coordinates}", "\text{matrix } N(\zeta, M)"'] & \text{New theta coordinates } (\theta'^A_i)_i
\end{tikzcd}$$

*ζ is a primitive 4-th root of unity given by the Weil-pairings of symplectic basis.

The right choice of theta structure propagates

When there is only one gluing isogeny, only 2 change of theta structures are needed

$$A_0 = A \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \quad \cdots \quad A_{e-1} \xrightarrow{f_e} A_e = B$$

Change of theta coordinates

Change of theta coordinates

Evaluating a gluing 2-isogeny

Issue 2:

- The evaluation algorithm:

$$(\theta_i^A(x))_i \xrightarrow{H} * \xrightarrow{S} * \xrightarrow{\star(1/\widetilde{\theta}_i^B(0_B))_i} * \xrightarrow{H} (\theta_i^B(f(x)))_i$$

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Solution 2: Using x and translates $x+T$ where $[2]T \in \ker(f)$, we can evaluate $f(x)$.

A dramatic improvement of SQIsign with dimension 2

Table: Comparison of time performance in ms of SQIsign (NIST round 1) and SQIsign (NIST round 2) on an Intel Core i5-1335U 4600MHz CPU.

		NIST I	NIST III	NIST V
SQIsign v 1.0	Key Gen.	355.72	5 625.72	22 445.3
	Signature	554.78	10 553.18	41 322.21
	Verification	7.77	195.86	571.77
SQIsign v 2.0	Key Gen.	10.63	32.05	51.37
	Signature	24.53	74.20	126.72
	Verification	1.13	4.10	8.49

PEGASIS: the proof dimension 4 can be efficient

Paper	Impl.	500	1000	1500	2000	4000
SCALLOP [FFK+23]*	C++	35s	12m30s	–	–	–
SCALLOP-HD [CLP24]*	Sage	88s	19m	–	–	–
PEARL-SCALLOP [ABE+24]	C++	30s	58s	12m	–	–
KLaPoTi [PPS24]	Sage	200s	–	–	–	–
	Rust	1.95s	–	–	–	–
PEGASIS (This work)	Sage	1.53s	4.21s	10.5s	21.3s	2m2s

Table: Comparison between PEGASIS and other effective group actions in the literature. The last 5 columns gives the timings corresponding to the different security levels, where s/m gives the number of seconds/minutes in wall-clock time. SCALLOP and SCALLOP-HD are starred because they were measured on a different hardware setup.

Thank you for listening

- SIDH attacks and HD isogenies are a breakthrough in isogeny based cryptography.
- There have been many grounbreaking constructive applications (*e.g.* SQIsign) and new applications are still unfolding.
- Research is still needed to accelerate HD algorithms (*e.g.* for odd degree).