Signing with higher dimensional isogenies

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Acknowledgements to Luca De Feo

12 October 2023

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The Deuring correspondence

The Deuring correspondence

Supersingular elliptic curves	Quaternions
$j(E)$ or $j(E)^p$ supersingular	$\mathcal{O} \cong \mathrm{End}(E)$ maximal order in $\mathcal{B}_{p,\infty}$
$\varphi : E \longrightarrow E'$	left $\mathcal{O}$-ideal and right $\mathcal{O}'$-ideal I_φ
$\varphi, \psi : E \longrightarrow E'$	$I_\varphi \sim I_\psi$ $(I_\psi = I_\varphi \alpha)$
$\widehat{\varphi}$	$\overline{I_\varphi}$
$\varphi \circ \psi$	$I_\psi \cdot I_\varphi$
$\theta \in \mathrm{End}(E)$	Principal ideal $\mathcal{O}\theta$
$\deg(\varphi)$	$\mathrm{nrd}(I_\varphi)$

Hard and easy problems

Hard problems

- **Supersingular Isogeny Problem:** Given two supersingular elliptic curves $E_1, E_2/\mathbb{F}_{p^2}$, find an isogeny $\varphi : E_1 \longrightarrow E_2$.
- **Supersingular End Ring Problem:** Given a supersingular elliptic curve $E/\mathbb{F}_{p^2}$, compute $\mathrm{End}(E)$.
- These two problems are equivalent [Wes22].

Easy problems

- **Connecting ideal:** Given two maximal orders $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{B}_{p,\infty}$, find a left $\mathcal{O}_1$-ideal I that is also a right $\mathcal{O}_2$-ideal.
- **Vélu**: Given $G = \ker(\varphi)$ (with $\#G$ smooth), compute φ [Vél71].
- **Quaternion path problem:** Given a left $\mathcal{O}$-ideal I, find $J \sim I$ of smooth norm [KLPT14].
- **Ideal translation:** Given a left $\mathcal{O}$-ideal I of smooth norm, compute the associated isogeny φ_I [DKLPW20].

Computing isogenies via the Deuring correspondence

- Let E_1 and E_2 of known endomorphism rings $\mathcal{O}_1 \cong \mathrm{End}(E_1)$ and $\mathcal{O}_2 \cong \mathrm{End}(E_2)$.
- Compute a connecting ideal I between $\mathcal{O}_1$ and $\mathcal{O}_2$.
- Compute $J \sim I$ of smooth norm via KLPT.
- Translate J into an isogeny $\varphi_J : E_1 \longrightarrow E_2$.

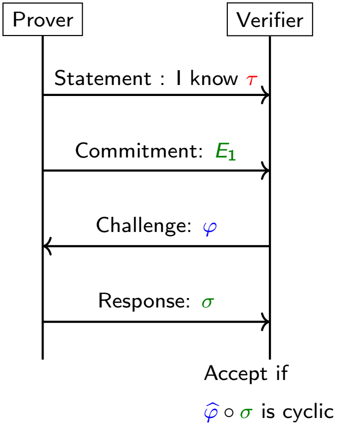
Becomes hard when $\mathrm{End}(E_1)$ or $\mathrm{End}(E_2)$ is unknown.

SQIsign and effective Deuring correspondence

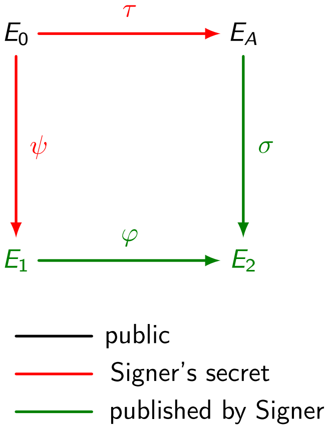
The SQIsign identification scheme [DKLPW20; FLLW23]

$$\begin{tikzcd} E_0 \arrow[r, "\tau"] \arrow[d, "\psi"'] & E_A \arrow[d, "\sigma"] \\ E_1 \arrow[r, "\varphi"] & E_2 \end{tikzcd}$$

- public
- Prover's secret
- published by Verifier
- published by Prover



Fiat-Shamir transform [FS87]



Signature: message m, public key E_A, secret key τ.

- Commitment $\psi : E_0 \longrightarrow E_1$.
- Challenge $\varphi := H(E_1, m)$ (where H is a hash function).
- Compute and send signature (E_1, σ) to the verifier.

Verification: $E_A, m, (E_1, \sigma)$.

- Recompute $\varphi := H(E_1, m)$.
- Verify that $\widehat{\varphi} \circ \sigma$ is cyclic.

How to compute the signature σ ?

$$
\begin{tikzcd}
E_0 \arrow[r, "\tau", "I_\tau"'] \arrow[d, "I_\psi"', "\psi"] & E_A \arrow[d, dashed, "I"', "\sigma"] \\
E_1 \arrow[r, "\varphi", "I_\varphi"'] & E_2
\end{tikzcd}
$$

- Compute $J := \overline{I_\tau} \cdot I_\psi \cdot I_\varphi$.
- Find $I \sim J$ random of norm $\mathrm{nrd}(I) = \ell^e$ (KLPT).
- Compute σ associated to I.

Ideal-to-isogeny I [GPS20]

Input: $E/\mathbb{F}_{p^2}$ supersingular, $\mathcal{O} \cong \mathrm{End}(E)$ and I a left $\mathcal{O}$-ideal of smooth norm.

Output: $\varphi_I : E \longrightarrow E_I$.

- Compute
$$E[I] := \{P \in E \mid \forall \alpha \in I, \quad \alpha(P) = 0\}.$$
- Compute φ_I of kernel $E[I]$ in $O(\mathrm{poly}(\max_{\ell | \mathrm{nrd}(I)} \ell))$ operations over $\mathbb{F}_{p^k}$, where $E[I] \subseteq E(\mathbb{F}_{p^k})$.

Issue: If I is a KLPT output, then $\mathrm{nrd}(I) \simeq p^{15/4} \gg p$ so k is exponentially big. Not practical for SQIsign !

Ideal-to-isogeny II [FLLW23]

Main idea: Cut the computation into smaller pieces. Write

$$I = I_0 \cdot I_1 \cdots I_{n-1} \quad \text{and} \quad \varphi_I = \varphi_{n-1} \circ \cdots \circ \varphi_1 \circ \varphi_0$$

with $\mathrm{nrd}(I_0) = \cdots = \mathrm{nrd}(I_{n-1}) = \ell^f$.

$$\begin{tikzcd}
E = E_0 \arrow[r, "I_0", "\varphi_0"'] & E_1 \arrow[loop above, "\theta_1"] \arrow[r, "I_1", "\varphi_1"'] & E_2 \arrow[loop above, "\theta_2"] & \cdots & E_{n-1} \arrow[loop above, "\theta_{n-1}"] \arrow[r, "I_{n-1}", "\varphi_{n-1}"'] & E_n = E_I
\end{tikzcd}$$

The endomorphisms θ_i are meant to refresh the ℓ^f-torsion.

Torsion requirements: $\ell^f T | p^2 - 1$ so that $E[\ell^f T] \subseteq E(\mathbb{F}_{p^4})$, where $\deg(\theta_i) = T^2$ and $T \simeq p^{5/4}$.

Issue: This is slow!

Higher dimensional isogenies

Another method to compute σ [DLRW23]

Issue in SQIsign: $\deg(\sigma)$ has to be smooth $\deg(\sigma) = \ell^e \simeq p^{15/4}$.

Our idea: Take $\deg(\sigma)$ non smooth. Then $\deg(\sigma) \simeq \sqrt{p}$.

- Evaluate σ on $E_A[\ell^e] \subseteq E_A(\mathbb{F}_{p^2})$.
- Use the following algorithm to evaluate σ everywhere.

Theorem (Robert, 2022)

Let $\sigma : E \longrightarrow E'$ of degree $q < \ell^e$. There exists a polynomial time algorithm with:

- ***Input:*** $(\sigma(P_1), \sigma(P_2))$, *where* (P_1, P_2) *is a basis of* $E[\ell^e]$ *and* $Q \in E(\mathbb{F}_{p^2})$.
- ***Output:*** $\sigma(Q)$.

Context: This idea comes from the attacks against SIDH [CD23; MM22; Rob23].

Evaluating σ

Main idea: Use the alternate path $\varphi \circ \psi \circ \widehat{\tau}$.

$$\begin{tikzcd} E_0 \arrow[r, "\tau", "I_\tau"'] \arrow[d, "\psi"', "I_\psi"] & E_A \arrow[d, dashed, "I"', "\sigma"] \\ E_1 \arrow[r, "\varphi", "I_\varphi"'] & E_2 \end{tikzcd}$$

- Let $\gamma := \widehat{\psi} \circ \widehat{\varphi} \circ \sigma \circ \tau \in \mathrm{End}(E_0)$.
- We have $\mathcal{O}_0\gamma = I_\tau \cdot I \cdot \overline{I_\varphi} \cdot \overline{I_\psi}$ so we can compute γ.
- Then:
$$[D_\psi D_\varphi D_\tau]\sigma = \varphi \circ \psi \circ \gamma \circ \widehat{\tau}$$
- We can evaluate σ on $P \in E_A[\ell^e]$ provided $(D_\psi D_\varphi D_\tau) \wedge \ell = 1$:
$$\sigma(P) = [\lambda]\varphi \circ \psi \circ \gamma \circ \widehat{\tau}(P),$$
with $\lambda D_\psi D_\varphi D_\tau \equiv 1 \mod \ell^e$.

d-isogenies and the dual isogeny in higher dimension

Definition (d-isogeny)

Let $\varphi : (A, \lambda_A) \longrightarrow (B, \lambda_B)$ be an isogeny between two principally polarized abelian varieties (PPAV). We define:

- $\widetilde{\varphi} := \lambda_A^{-1} \circ \widehat{\varphi} \circ \lambda_B : B \longrightarrow A$.

$$B \xrightarrow{\lambda_B} \widehat{B} \xrightarrow{\widehat{\varphi}} \widehat{A} \xrightarrow{\lambda_A^{-1}} A$$

- We say that φ is a <u>d-isogeny</u> if $\widetilde{\varphi} \circ \varphi = [d]_A$.

Kani's embedding lemma [Kan97]

Definition (isogeny diamond)

An (a, b)-isogeny diamond is a commutative diagram s.t.:

$$\begin{tikzcd} A' \arrow[r, "\varphi'"] & B' \\ A \arrow[u, "\psi"] \arrow[r, "\varphi"] & B \arrow[u, "\psi'"'] \end{tikzcd}$$

where φ, φ' are a-isogenies and ψ, ψ' are b-isogenies.

Lemma (Kani)

Consider the (a, b)-isogeny diamond on the left. Then:

- $F : A \times B' \longrightarrow B \times A'$,

$$F := \begin{pmatrix} \varphi & \widetilde{\psi}' \\ -\psi & \widetilde{\varphi}' \end{pmatrix}$$

is a d-isogeny with $d = a + b$.

- *If $a \wedge b = 1$, then*

$$\ker(F) = \{(\widetilde{\varphi}(x), \psi'(x)) \mid x \in B[d]\}.$$

Application of Kani's lemma to SQIsignHD

Embedding σ in higher dimension:

- Let $q = \deg(\sigma)$.
- Let $a_1, a_2 \in \mathbb{Z}$ s.t. $a_1^2 + a_2^2 + q = \ell^e$.
- q should be good: $\ell^e - q$ prime $\equiv 1 \mod 4$.
- Consider the isogeny diamond:

$$\begin{tikzcd} E_2^2 \arrow[r, "\alpha_2"] & E_2^2 \\ E_A^2 \arrow[u, "\Sigma"] \arrow[r, "\alpha_A"] & E_A^2 \arrow[u, "\Sigma"'] \end{tikzcd}$$

where $\Sigma := \mathrm{Diag}(\sigma, \sigma)$ and for $i = A, 2$,

$$\alpha_i := \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix} \in \mathrm{End}(E_i^2).$$

Application of Kani's lemma to SQIsignHD

Embedding σ in higher dimension:

- Then
$$F := \begin{pmatrix} \alpha_1 & \widetilde{\Sigma} \\ -\Sigma & \widetilde{\alpha}_A \end{pmatrix} \in \mathrm{End}(E_A^2 \times E_2^2).$$
is an ℓ^e-isogeny.
- And
$$\ker(F) = \{([a_1]R - [a_2]S, [a_2]R + [a_1]S, \sigma(R), \sigma(S)) \mid R, S \in E_A[\ell^e]\}.$$
- F can be computed in polynomial time [LR12; LR15; LR23; DLRW23].

Algorithm for higher dimensional isogeny computations

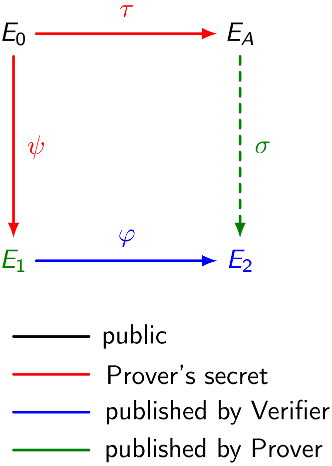
- The ℓ^e-isogeny F can be computed as a chain of ℓ-isogenies:

$$\mathcal{A}_0 \xrightarrow{F_0} \mathcal{A}_1 \xrightarrow{F_2} \mathcal{A}_2 \quad \cdots \quad \mathcal{A}_{e-1} \xrightarrow{F_e} \mathcal{A}_e$$

- Each ℓ-isogeny can be computed in $O(\ell^g)$ efficiently in the Θ-model [LR12; LR15; LR23; DLRW23].
- The whole chain can be computed in time $O(\ell^g e \log(e))$ [JD11; DLRW23].
- This method is valid in any dimension g.

SQIsignHD: signing with higher dimensional isogenies

SQIsignHD identification scheme [DLRW23]



Secret key: τ

Commitment: E_1

Challenge: φ

Response: $(q, \sigma(P_1), \sigma(P_2))$

- Compute $I \sim \overline{I_\tau} \cdot I_\psi \cdot I_\varphi$ random of norm $q \simeq \sqrt{p}$.
- Compute a canonical basis (P_1, P_2) of $E_A[\ell^e]$.
- Evaluate $\sigma = \varphi_I$ on (P_1, P_2).
- Send $(q, \sigma(P_1), \sigma(P_2))$.

Very fast !

SQIsignHD identification scheme [DLRW23]

$$\begin{tikzcd} E_0 \arrow[r, "\tau"] \arrow[d, "\psi"'] & E_A \arrow[d, dashed, "\sigma"] \\ E_1 \arrow[r, "\varphi"] & E_2 \end{tikzcd}$$

public

Prover's secret

published by Verifier

published by Prover

Response: $(q, \sigma(P_1), \sigma(P_2))$

Verification: Compute the embedding $F \in \mathrm{End}(E_A^2 \times E_2^2)$ of σ.

- Find $a_1, a_2 \in \mathbb{Z}$ such that $a_1^2 + a_2^2 + q = \ell^e$ (Cornacchia).
- Compute the canonical basis (P_1, P_2) of $E_A[\ell^e]$.
- Compute $\ker(F)$, knowing $a_1, a_2, P_1, P_2, \sigma(P_1), \sigma(P_2)$.
- Compute F.
- Accept if $F \in \mathrm{End}(E_A^2 \times E_2^2)$ and $F(Q, 0, 0, 0) = ([a_1]Q, -[a_2]Q, *, 0)$.

Proof of concept.

Fiat-Shamir transform [FS87] of SQIsignHD

$$\begin{tikzcd}
E_0 \arrow[r, "\tau"] \arrow[d, "\psi"'] & E_A \arrow[d, dashed, "\sigma"] \\
E_1 \arrow[r, "\varphi"] & E_2
\end{tikzcd}$$

public

Signer's secret

published by Signer

Signature: message m, public key E_A, secret key τ.

- Commitment $\psi : E_0 \longrightarrow E_1$.
- Challenge $\varphi := H(E_1, m)$ (where H is a hash function).
- Compute and send signature $(E_1, q, \sigma(P_1), \sigma(P_2))$ to the verifier.

Verification: $E_A, m, (E_1, q, \sigma(P_1), \sigma(P_2))$.

- Recompute $\varphi := H(E_1, m)$.
- Use $\varphi, q, \sigma(P_1), \sigma(P_2)$ to compute the embedding F of σ.
- Check that $F \in \mathrm{End}(E_A^2 \times E_2^2)$ and $F(Q, 0, 0, 0) = ([a_1]Q, -[a_2]Q, *, 0)$.

Parameter choice

- **Characteristic:** as in SIDH,

$$p = c\ell^f \ell'^{f'} - 1.$$

  with c small.

- In practice $\ell = 2$ and $\ell' = 3$. For NIST-1, $p = 13 \cdot 2^{126} \cdot 3^{78} - 1$.
- **Use of ℓ^f-torsion:** the 4-dimensional isogeny F.
- **Use of $\ell'^{f'}$-torsion:** τ, ψ, φ.
- $E_0 : y^2 = x^3 + x$ defined over $\mathbb{F}_p$ ($p \equiv 3 \mod 4$). $\mathrm{End}(E_0)$ is known.

Outline of the security analysis

Theorem (Fiat-Shamir, 1986)

Let ID be an identification protocol that is:

- ***Complete:*** *a honest execution is always accepted by the verifier.*
- ***Sound:*** *an attacker cannot "guess" a response.*
- ***Zero-knowledge:*** *the response does not leak any information on the secret key.*

Then the Fiat-Shamir transform of ID is a universally unforgeable signature under chosen message attacks in the random oracle model.

Soundness

Similar to SQIsign

Proposition (Special soundness)

Given two transcripts $(E_1, \varphi, q, \sigma(P_1), \sigma(P_2)), (E_1, \varphi', q', \sigma'(P_1'), \sigma'(P_2'))$ *with the same commitment* E_1 *and* $\varphi \neq \varphi'$, *we can extract* $\alpha \in \mathrm{End}(E_A)$ *non-scalar.*

Proof.

- Exctract σ from $(q, \sigma(P_1), \sigma(P_2))$ and σ' from $(q', \sigma'(P_1'), \sigma'(P_2'))$.
- Then $\alpha := \widehat{\sigma'} \circ \varphi' \circ \widehat{\varphi} \circ \sigma \in \mathrm{End}(E_A)$ is non-scalar.

$\square$

Zero-knowledge

Definition (Recall)

We say that an integer q is **good** if $\ell^e - q$ is a prime $\equiv 1 \mod 4$.

Definition (RUGDIO)

A random uniform good degree isogeny oracle (RUGDIO):
Input: A supersingular elliptic curve $E/\mathbb{F}_{p^2}$.
Output: An isogeny $\sigma : E \longrightarrow E'$ of good degree q s.t.

- E' is uniform among supersingular elliptic curves.
- Given E', σ is uniform among isogenies of good degree $E \longrightarrow E'$.

Zero-knowledge

Theorem

Assume that:

- *E_1 is computationally close to uniform.*
- *We have access to a RUGDIO.*

Then SQIsignHD is computationally honest-verifier zero-knowledge.

Proof.

We build a simulator $\mathcal{S}$ of protocol transcripts:

- $\mathcal{S}$ calls the RUGDIO to generate $(q, \sigma(P_1), \sigma(P_2))$.
- $\mathcal{S}$ generates a random challenge $\widehat{\varphi} : E_2 \longrightarrow E_1$.
- $\mathcal{S}$ outputs $(E_1, \varphi, q, \sigma(P_1), \sigma(P_2))$.

$\square$

Zero-knowledge: comparison with SQIsign

Heuristic assumptions to prove the zero-knowledge property

In SQIsign:

- $\sigma : E_1 \longrightarrow E_2$ is computationally indistinguishable from a random isogeny of degree ℓ^e.

In SQIsignHD:

- E_1 is computationally close to uniform.
- We have access to a RUGDIO.

Fast and compact signatures

- **Signature time:** 28 ms on a 13th Gen Intel(R) Core(TM) i5-1335U (4600MHz) CPU.

Signature size comparison

	In SQIsign	In SQIsignHD
Asymptotic (in bits)	$\sim 23/4 \log_2(p)$	$\sim 13/4 \log_2(p)$
NIST-1 security level (in bytes)	204	109

A promising POC for the verification

Timing: 855 ms in `sagemath` with $p = 13 \cdot 2^{126} \cdot 3^{78} - 1$ on a 13th Gen Intel(R) Core(TM) i5-1335U (4600MHz) CPU.

- **Challenge computation (φ):** 60 ms.
- **Dimension 4 2^{142}-isogeny:** 770 ms with Θ-coordinates of level 2.
  - $F \in \mathrm{End}(E_A^2 \times E_2^2)$ is divided in two:

$$E_A^2 \times E_2^2 \xrightarrow{F_1} \mathcal{C} \xleftarrow{\widetilde{F}_2} E_A^2 \times E_2^2$$

  - Compute F_1 and $\widetilde{F}_2$.
  - Check that codomains of F_1 and $\widetilde{F}_2$ match.
  - Compute $F_2 := \widetilde{\widetilde{F}}_2$
- **Isogeny evaluation:** 25 ms.

$$F(Q,0,0,0) = F_2 \circ F_1(Q,0,0,0)$$

Conclusion

Comparison of SQIsignHD with SQIsign

	SQIsign	SQIsignHD
Security	✗ Ad-hoc heuristic: • Distribution of σ.	✓ Simpler heuristics: • RUGDIO; • Distribution of E_1.
Signing time	✗ 400 ms for NIST-1	✓ 28 ms for NIST-1
Signature size	✓ 204 bytes for NIST-1	✓ 109 bytes for NIST-1
Verification	✓ Fast (6 ms for NIST-1)	✗ 850 ms for NIST-1 in `sagemath`

Thank you for listening.

Find our pre-print here: https://eprint.iacr.org/2023/436