SQISignHD: signing with higher dimensional isogenies

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Joint work with Antonin Leroux, Damien Robert and Benjamin Wesolowski
Acknowledgements to Luca De Feo

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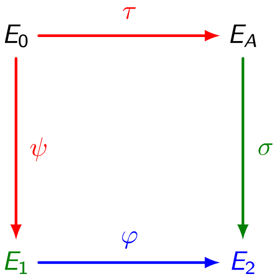
SQISign

The SQISign identification scheme (DFKLPW, 2020)

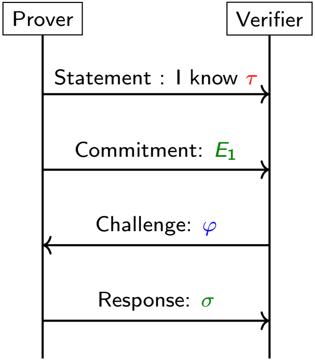


public

Prover's secret

published by Verifier

published by Prover



Accept if

$\widehat{\varphi} \circ \sigma$ is cyclic

Fiat-shamir transform

$$
\begin{tikzcd}
E_0 \arrow[r, "\tau"] \arrow[d, "\psi"'] & E_A \arrow[d, "\sigma"] \\
E_1 \arrow[r, "\varphi"] & E_2
\end{tikzcd}
$$

public

Signer's secret

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Signature: message m, public key E_A, secret key τ.

- Commitment $\psi : E_0 \longrightarrow E_1$.
- Challenge $\varphi := H(E_1, m)$ (where H is a hash function).
- Compute and send signature (E_1, σ) to the verifier.

Fiat-shamir transform

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Verification: $E_A, m, (E_1, \sigma)$.

- Recompute $\varphi := H(E_1, m)$.
- Verify that $\widehat{\varphi} \circ \sigma$ is cyclic.

NB: Not necessary to send E_1.

How to compute the signature σ ?

The Deuring correspondence

Supersingular elliptic curves	Quaternions
$j(E)$ or $j(E)^p$ supersingular	$\mathcal{O} \cong \mathrm{End}(E)$ maximal order in $\mathcal{B}_{p,\infty}$
$\varphi : E \longrightarrow E'$	left $\mathcal{O}$-ideal and right $\mathcal{O}'$-ideal I_φ
$\varphi, \psi : E \longrightarrow E'$	$I_\varphi \sim I_\psi$ $(\exists \alpha \in \mathcal{B}_{p,\infty},\ I_\psi = I_\varphi \alpha)$
$\widehat{\varphi}$	$\overline{I_\varphi}$
$\varphi \circ \psi$	$I_\psi \cdot I_\varphi$
$\theta \in \mathrm{End}(E)$	Principal ideal $\mathcal{O}\theta$
$\deg(\varphi)$	$\mathrm{nrd}(I_\varphi)$

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$$\begin{tikzcd}
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- Compute $J := \overline{I_\tau} \cdot I_\psi \cdot I_\varphi$.
- Find $I \sim J$ of norm $\mathrm{nrd}(I) = \ell^e$ (KLPT, 2014 & FKLPW, 2020).
- Compute σ associated to I.

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Ideal-to-isogeny translation:

- $\mathrm{nrd}(I) = \deg(\sigma) = p^{15/4}$ (DFLW, 2022).
- Not enough $\ell^\bullet$-torsion accessible.
- σ is computed piecewise (DFLW, 2022). **It is slow.**

Representing an isogeny in higher dimension

Efficient isogeny representation

Definition

An **efficient representation** of an isogeny $\varphi : E \longrightarrow E'$ is a couple $(D, \mathscr{A})$, where:

1. D is polynomial size data (in $\log(p)$, $\log(\deg(\varphi))$).
2. $\mathscr{A}$ is an algorithm computing $\varphi(P)$ on input D and $P \in E(\mathbb{F}_{p^k})$ in polynomial time (in $\log(p)$, $\log(\deg(\varphi))$ and k).

Efficient isogeny representation

Example (efficient representation): If $\varphi : E \longrightarrow E'$ is an ℓ^e-isogeny:

- D: chain of ℓ-isogenies $E = E_0 \xrightarrow{\varphi_0} E_1 \cdots E_{e-1} \xrightarrow{\varphi_{e-1}} E_e = E'$.
- $\mathscr{A}$: evaluate each isogeny of the chain successively.

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State of the art :

- All efficient representations are essentially equivalent to this one.
- Only smooth degree isogenies can be represented (explains the use of KLPT in SQISign).
- In SQISign, the conversion of an ideal I into such a representation (isogeny-chain) is costly.

d-isogenies

- Let $\varphi : (A, \lambda_A) \longrightarrow (B, \lambda_B)$ be an isogeny between PPAV[1].
- Consider $\widetilde{\varphi} : (B, \lambda_A) \longrightarrow (A, \lambda_B)$ the isogeny

$$B \xrightarrow{\lambda_B} \widehat{B} \xrightarrow{\widehat{\varphi}} \widehat{A} \xrightarrow{\lambda_A^{-1}} A$$

[1] Principally polarized abelian varieties.

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Definition

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Lemma

If $d = \ell^e$, then φ can be decomposed as a product of ℓ-isogenies

$$A = A_0 \xrightarrow{\varphi_0} A_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{e-2}} A_{e-1} \xrightarrow{\varphi_{e-1}} A_e = B$$

[1] Principally polarized abelian varieties.

Kani's lemma (K, 1997)

An (a, b)-**isogeny diamond** is a commutative diagram:

$$\begin{tikzcd} A' \arrow[r, "\varphi'"] & B' \\ A \arrow[u, "\psi"] \arrow[r, "\varphi"] & B \arrow[u, "\psi'"'] \end{tikzcd}$$

with:

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Lemma (K, 1997)

(i) *Assume a and b coprime with p. Then*

$$F := \begin{pmatrix} \varphi & \widetilde{\psi'} \\ -\psi & \widetilde{\varphi'} \end{pmatrix} : A \times B' \longrightarrow B \times A'$$

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is a d-isogeny (with $d := a + b$).
(ii) *If $a \wedge b = 1$, we have*

$$\ker(F) = \{(\widetilde{\varphi}(x), \psi'(x)) \mid x \in B[d]\}.$$

Representation in dimension 4

We can now represent isogenies of non-smooth degrees!

Let:

- $\sigma : E_A \longrightarrow E_2$ of degree $q < \ell^e$.
- $\Sigma := \mathrm{Diag}(\sigma, \sigma) : E_A^2 \longrightarrow E_2^2$.
- $a_1, a_2 \in \mathbb{Z}$, s.t.

$$a_1^2 + a_2^2 + q = \ell^e.$$

- $\alpha \in \mathrm{End}(E_A^2)$ given by:

$$\alpha := \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix}$$

- and $\alpha' \in \mathrm{End}(E_2^2)$ defined as α.

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The $(q, a_1^2 + a_2^2)$-isogeny diamond

$$\begin{tikzcd} E_2^2 \arrow[r, "\alpha'"] & E_2^2 \\ E_A^2 \arrow[u, "\Sigma"] \arrow[r, "\alpha"] & E_A^2 \arrow[u, "\Sigma"'] \end{tikzcd}$$

yields an ℓ^e-isogeny

$$F := \begin{pmatrix} \alpha & \widetilde{\Sigma} \\ -\Sigma & \widetilde{\alpha'} \end{pmatrix} \in \mathrm{End}(E_A^2 \times E_2^2).$$

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$$F := \begin{pmatrix} \alpha & \widetilde{\Sigma} \\ -\Sigma & \widetilde{\alpha'} \end{pmatrix} \in \mathrm{End}(E_A^2 \times E_2^2).$$

- We have
$$\ker(F) = \{(\widetilde{\alpha}(P), \Sigma(P)) \mid P \in E_A^2[\ell^e]\}.$$
- It suffices to compute $\sigma(E_A[\ell^e])$ to compute $\ker(F)$.
- F can then be computed as a chain of ℓ-isogenies.
- Knowing F, we can evaluate σ everywhere.

Representation in dimension 4

The bad news: We have to compute isogenies in dimension 4.

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The good news:

- Much more freedom on $q = \deg(\sigma)$.
- Smaller degree $q = \mathrm{nrd}(I) \simeq \sqrt{p}$.
- Recall that $q = \ell^e \simeq p^{15/4}$ in SQISign.

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- Recall that $q = \ell^e \simeq p^{15/4}$ in SQISign.

How much freedom on the choice of q?

- Constraint: we can find $a_1, a_2 \in \mathbb{Z}$ such that $a_1^2 + a_2^2 + q = \ell^e$.
- In practice: $\ell^e - q$ is a prime $\equiv 1 \mod 4$.

Representation in dimension 8

Representing σ of any degree $q < \ell^e$

- Find $a_1, a_2, a_3, a_4 \in \mathbb{Z}$ such that

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + q = \ell^e.$$

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$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + q = \ell^e.$$

- Let $\Sigma := \mathrm{Diag}(\sigma, \cdots, \sigma) : E_A^4 \longrightarrow E_2^4$,

$$\alpha := \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & -a_2 & a_1 \end{pmatrix} \in \mathrm{End}(E_A^4)$$

  and $\alpha' \in \mathrm{End}(E_2^4)$ defined as α.

- Instead, we can represent σ with the ℓ^e-isogeny

$$F := \begin{pmatrix} \alpha & \widetilde{\Sigma} \\ -\Sigma & \widetilde{\alpha}' \end{pmatrix} \in \mathrm{End}(E_A^4 \times E_2^4).$$

Algorithms for response and verification

The response algorithm

Overview of the response algorithm

$$\begin{tikzcd}[column sep=huge, row sep=huge]
E_0 \arrow[r, "\tau", "I_\tau"'] \arrow[d, shift right=3, "\psi' \; I_{\psi'}"'] \arrow[d, shift left=3, "\psi \; I_\psi"] & E_A \arrow[d, dashed, "I"', "\sigma"] \\
E_1 \arrow[r, "\varphi", "I_\varphi"'] & E_2
\end{tikzcd}$$

- Compute $J := \overline{I_\tau} \cdot I_\psi \cdot I_\varphi$.
- Find $I \sim J$ random of norm $q < \ell^e$ s.t. $\ell^e - q$ is prime $\equiv 1 \mod 4$.

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\end{tikzcd}$$

- Compute $J := \overline{I_\tau} \cdot I_\psi \cdot I_\varphi$.
- Find $I \sim J$ <u>random</u> of norm $q < \ell^e$ s.t. $\ell^e - q$ is prime $\equiv 1 \mod 4$.
- Generate (P_1, P_2) a canonical basis of $E_A[\ell^e]$.
- Compute $(\sigma(P_1), \sigma(P_2))$ using $\varphi \circ \psi \circ \widehat{\tau}$.
- Send $(q, \sigma(P_1), \sigma(P_2))$.

The response algorithm

Evaluating σ

$$\begin{tikzcd}
E_0 \arrow[rr, "\tau", "I_\tau"'] \arrow[d, shift right=3, "I_{\psi'}"', "\psi'"] \arrow[d, "\psi", "I_\psi"'] & & E_A \arrow[d, dashed, "I"', "\sigma"] \\
E_1 \arrow[rr, "\varphi", "I_\varphi"'] & & E_2
\end{tikzcd}$$

- Let $\gamma := \widehat{\psi} \circ \widehat{\varphi} \circ \sigma \circ \tau \in \mathrm{End}(E_0)$.
- We have $\mathcal{O}_0\gamma = I_\tau \cdot I \cdot \overline{I_\varphi} \cdot \overline{I_\psi}$ so we can compute γ.

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E_0 \arrow[r, "\tau", "I_\tau"'] \arrow[d, shift right=2, "\psi'"', "I_{\psi'}"'] \arrow[d, shift left=2, "\psi", "I_\psi"] & E_A \arrow[d, dashed, "I"', "\sigma"] \\
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- We have $\mathcal{O}_0\gamma = I_\tau \cdot I \cdot \overline{I_\varphi} \cdot \overline{I_\psi}$ so we can compute γ.
- Then:

$$[D_\psi D_\varphi D_\tau]\sigma = \varphi \circ \psi \circ \gamma \circ \widehat{\tau}$$

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E_0 \arrow[r, "\tau", "I_\tau"'] \arrow[d, shift right=2, "\psi' \; I_{\psi'}"'] \arrow[d, shift left=2, "\psi \; I_\psi"] & E_A \arrow[d, dashed, "\sigma", "I"'] \\
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- Then:

$$[D_\psi D_\varphi D_\tau]\sigma = \varphi \circ \psi \circ \gamma \circ \widehat{\tau}$$

- We can evaluate σ on $P \in E_A[\ell^e]$ provided $(D_\psi D_\varphi D_\tau) \wedge \ell = 1$:

$$\sigma(P) = [\lambda]\varphi \circ \psi \circ \gamma \circ \widehat{\tau}(P),$$

with $\lambda D_\psi D_\varphi D_\tau \equiv 1 \mod \ell^e$.

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with $\lambda D_\psi D_\varphi D_\tau \equiv 1 \mod \ell^e$.

- In SQISignHD, D_ψ, D_φ and D_τ are powers of a prime $\ell' \neq \ell$.

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- We know a basis $\mathcal{B}_0$ of $\mathcal{O}_0 \cong \mathrm{End}(E_0)$.
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Why do we need two paths $\psi, \psi' : E_0 \longrightarrow E_1$? Because $D_\psi = \ell'^\bullet$ is not coprime with $D_\varphi = \ell'^\bullet$.

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Step 2: Evaluate $\mathcal{B}_1$ on $\ker(\varphi)$ and solve DLPs to find a basis of I_φ.

- Let $\mathcal{B}_1 := (\beta_1, \cdots, \beta_4)$ and $\ker(\varphi) := \langle P \rangle$ (assuming φ cyclic).
- Compute $\beta_i(P)$ for $1 \leq i \leq 4$.
- Find i, j such that $(\beta_i(P), \beta_j(P))$ generate $E_1[D_\varphi]$.
- Let $k \neq i, j$. Find $a, b \in \mathbb{Z}/D_\varphi\mathbb{Z}$ s.t. $\beta_k(P) = a\beta_i(P) + b\beta_j(P)$.
- Let $\gamma := \beta_k - a\beta_i - b\beta_j$ and return $I_\varphi := \mathcal{O}_1\gamma + \mathcal{O}_1 D_\varphi$.

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Example: $p = 2^{128}3^{81} - 1$ for NIST-1 level of security.

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E_1 \arrow[r, "\varphi", "D_\varphi"'] & E_2
\end{tikzcd}$$

Degree choices:

- $D_\psi = D_\tau = \ell'^{2f'} \simeq p$.
- $D_{\psi'} = \ell^{2f} \simeq p$.
- $D_\varphi = \ell'^{f'} \simeq \sqrt{p}$.

The verification algorithm

Notations:

$$\begin{tikzcd} E_2^2 \arrow[r, "\alpha'"] & E_2^2 \\ E_A^2 \arrow[u, "\Sigma"] \arrow[r, "\alpha"] & E_A^2 \arrow[u, "\Sigma"'] \end{tikzcd}$$

$$\alpha := \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix}$$

and idem for α'. $\Sigma := \mathrm{Diag}(\sigma, \sigma)$

$$F := \begin{pmatrix} \alpha & \widetilde{\Sigma} \\ -\Sigma & \widetilde{\alpha}' \end{pmatrix}$$

Entry: $(q, \sigma(P_1), \sigma(P_2))$.

The verification algorithm

Notations:

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$$\mathcal{A}_0 \xrightarrow{F_1} \mathcal{A}_1 \cdots \mathcal{A}_{e-1} \xrightarrow{F_e} \mathcal{A}_e$$

with $\mathcal{A}_0 := E_2^2 \times E_A^2$.

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- Accept if $\mathcal{A}_e = \mathcal{A}_0$.

Verifying with less torsion

- We can divide F into

$$\mathcal{A} \xrightarrow{F_1} \mathcal{B} \xrightarrow{F_2} \mathcal{A}$$

where $\mathcal{A} := E_2^2 \times E_A^2$, F_i is an ℓ^{e_i}-isogeny (for $i = 1, 2$) and $e := e_1 + e_2$.

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- Knowing $(P_1, P_2, \sigma(P_1), \sigma(P_2))$ is sufficient to compute F_1 and $\widetilde{F_2}$.
- The verifier accepts if the codomains of F_1 and $\widetilde{F_2}$ match.

Verifying with less torsion

Advantages:

- Use ℓ^{f_1}-torsion with $f_1 \simeq e/2$ instead of ℓ^e-torsion ($f_1 \leq f$).
- $q < \ell^e$ is not constained by the accessible torsion (more freedom on the choice of I).
- Makes signature communications $\sigma(P_1), \sigma(P_2)$ twice more compact.

Higher dimensional isogeny computation

Goal: Compute an ℓ^e-isogeny $F : \mathcal{A} \longrightarrow \mathcal{B}$.

- Let $\mathscr{B}_0$ be a basis of $\ker(F)$.
- Decompose:

$$\mathcal{A} = \mathcal{A}_0 \xrightarrow{F_1} \mathcal{A}_1 \cdots \mathcal{A}_{e-1} \xrightarrow{F_e} \mathcal{A}_e = \mathcal{B}$$

- Let $\mathscr{B}_i := F_i \circ \cdots \circ F_1(\mathscr{B}_0)$.
- Then $\ker(F_i) = \langle [\ell^{e-i}]\mathscr{B}_{i-1} \rangle$.

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- Descend the computation tree to compute the F_i (DFJP, 11).

Computation tree for $e = 5$

$$\begin{tikzcd}
 & & & & \mathscr{B}_0 \arrow[dl] \arrow[dr, dashed, "F_1"] & & & & \\
 & & & [\ell]\mathscr{B}_0 \arrow[dl] \arrow[dr, dashed] & & \mathscr{B}_1 \arrow[dl] \arrow[dr, dashed, "F_2"] & & & \\
 & & [\ell^2]\mathscr{B}_0 \arrow[dl] \arrow[dr, dashed] & & [\ell]\mathscr{B}_1 \arrow[dl] \arrow[dr, dashed] & & \mathscr{B}_2 \arrow[dl] \arrow[dr, dashed, "F_3"] & & \\
 & [\ell^3]\mathscr{B}_0 \arrow[dl] \arrow[dr, dashed] & & [\ell^2]\mathscr{B}_1 \arrow[dl] \arrow[dr, dashed] & & [\ell]\mathscr{B}_2 \arrow[dl] \arrow[dr, dashed] & & \mathscr{B}_3 \arrow[dl] \arrow[dr, dashed, "F_4"] & \\
[\ell^4]\mathscr{B}_0 & & [\ell^3]\mathscr{B}_1 & & [\ell^2]\mathscr{B}_2 & & [\ell]\mathscr{B}_3 & & \mathscr{B}_4
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- Each F_i is computed with the Θ model (level $\ell = 2$).

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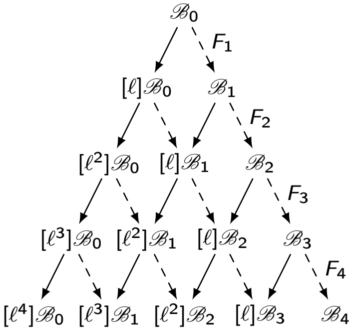

Commitment (and key generation)

Double path to the commitment

Goal: Compute $\psi, \psi' : E_0 \longrightarrow E_1$ and $I_\psi, I_{\psi'}$ with $\deg(\psi) = \ell^{2f}$ and $\deg(\psi) = \ell'^{2f'}$.

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- Compute $[\theta_1']_*\theta_2$ of kernel $\theta_1'(\ker \theta_2)$.
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- $I_\psi := \mathcal{O}_0\overline{\gamma} + \mathcal{O}_0\ell'^{2f'}$, $I_{\psi'} := \mathcal{O}_0\gamma + \mathcal{O}_0\ell^{2f}$.

Security analysis

Outline of the security analysis

Theorem (Fiat-Shamir, 1986)

Let ID be an identification protocol that is:

- ***Complete:*** *a honest execution is always accepted by the verifier.*
- ***Sound:*** *an attacker cannot "guess" a response.*
- ***Zero-knowledge:*** *the response does not leak any information on the secret key.*

Then the Fiat-Shamir transform of ID is a universally unforgeable signature under chosen message attacks in the random oracle model.

Soundness

Similar to SQISign

Proposition (Special soundness)

Assume $q \wedge \ell' = 1$.
Then given two transcripts $(E_1, \varphi, R), (E_1, \varphi', R')$ with the same commitment E_1 and $\varphi \neq \varphi'$, we can extract $\alpha \in \mathrm{End}(E_A)$ non-scalar.

Proof.

- Exctract σ from R and σ' from R'.
- Then $\alpha := \widehat{\varphi}' \circ \widehat{\sigma}' \circ \sigma \circ \varphi \in \mathrm{End}(E_A)$ is non-scalar.

□

Zero-knowledge

Definition

We say that an integer q is **good** if:

- $\ell^e - q$ is a prime $\equiv 1 \mod 4$.
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Zero-knowledge

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Definition (RUGDIO)

A random uniform good degree isogeny oracle (RUGDIO):
Input: A supersingular elliptic curve $E/\mathbb{F}_{p^2}$.
Output: An isogeny $\sigma : E \longrightarrow E'$ of good degree q s.t.

- E' is uniform in the supersingular isogeny graph.
- Given E', σ is uniform among isogenies of good degree $E \longrightarrow E'$.

Zero-knowledge

Theorem

Assume that:

- E_1 *is computationally close to uniform.*
- *We have access to a RUGDIO.*

Then SQISignHD is computationally honest-verifier zero-knowledge.

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Proof.

We build a simulator $\mathcal{S}$ of protocol transcripts:

- $\mathcal{S}$ calls the RUGDIO to generate an efficient representation R of $\sigma : E_A \longrightarrow E_2$.
- $\mathcal{S}$ generates a random challenge $\widehat{\varphi} : E_2 \longrightarrow E_1$.
- $\mathcal{S}$ outputs (E_1, φ, R).

$\square$

Zero-knowledge: comparison with SQISign

Heuristic assumptions to prove the zero-knowledge property

In SQISign:

- $\sigma : E_A \longrightarrow E_2$ is computationally indistinguishable from a random isogeny of degree ℓ^e.

In SQISignHD:

- E_1 is computationally close to uniform.
- We have access to a RUGDIO.

Zero-knowledge: the interest of dimension 8

In SQISign:

- $\sigma : E_A \longrightarrow E_2$ is computationally close to a random isogeny of degree ℓ^e.

Zero-knowledge: the interest of dimension 8

In SQISign:

- $\sigma : E_A \longrightarrow E_2$ is computationally close to a random isogeny of degree ℓ^e.

In RigorousSQISignHD:

- We have access to a RADIO.

Definition (RADIO)

A random any degree isogeny oracle (RADIO):
Input: A supersingular elliptic curve $E/\mathbb{F}_{p^2}$.
Output: An efficient representation of a uniformly random isogeny $\sigma : E \longrightarrow E'$ of degree $q < \ell^e$.

Performance

Compact signatures

In SQISign: $\sigma : E_A \longrightarrow E_2$ of degree $\ell^e \simeq p^{15/4}$.

In SQISignHD: $(j(E_1), q, \sigma(P_1), \sigma(P_2))$, where:

- (P_1, P_2) is a basis of $E_A[\ell^{f_1}]$.
  NB: no need to transmit (P_1, P_2).
- $\ell^{f_1} \simeq p^{1/4}$.
- $q \simeq \sqrt{p}$.
- $j(E_1) \in \mathbb{F}_{p^2}$ has size $2\log_2(p)$ bits.

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Signature size (in bits)

	In SQISign	In SQISignHD
In general	$\sim 15/4 \log_2(p)$	$\sim 13/4 \log_2(p)$
NIST-1 security level	1060	840

Fast signatures at the expense of verification

Fast signature:

- Only 1-dimensional isogenies: ψ, ψ', φ.
- Evaluating $(\sigma(P_1), \sigma(P_2))$ with $\varphi \circ \psi \circ \widehat{\tau}$ is fast.
- Preliminary implementation way faster than SQISign.

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Comparison with SQISign signatures:

- Slow ideal-to-isogeny translation: $I \longmapsto \sigma$.
- Piecewise computation involving 30 T-isogenies with $T \simeq p^{5/4}$.

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Verification:

- Isogenies in dimension 4 to compute.
- Known algorithms.
- But to be implemented...

Conclusion

Comparison of SQISignHD with SQISign

	SQISign	SQISignHD
Security	Ad-hoc heuristic: • Distribution of σ.	Simpler heuristics: • RUGDIO; • Distribution of E_1.

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Prime p	$\ell^f T \mid p^2 - 1$ with $T \simeq p^{5/4}$ • Slow isogeny computations • Not certain if it scales	$p = c\ell^f \ell'^{f'} - 1$ • Fast isogeny computations • Scales well

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Signature	σ with $\deg(\sigma) = \ell^e \simeq p^{15/4}$ • Ideal-to-isogeny translation • 30 T-isogenies involved	$(q, \sigma(P_1), \sigma(P_2))$ • Fast via $\varphi \circ \psi \circ \widehat{\tau}$

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	SQISign	SQISignHD
Security	Ad-hoc heuristic: • Distribution of σ.	Simpler heuristics: • RUGDIO; • Distribution of E_1.
Prime p	$\ell^f T \mid p^2-1$ with $T \simeq p^{5/4}$ • Slow isogeny computations • Not certain if it scales	$p = c\ell^f \ell'^{f'} - 1$ • Fast isogeny computations • Scales well
Signature	σ with $\deg(\sigma) = \ell^e \simeq p^{15/4}$ • Ideal-to-isogeny translation • 30 T-isogenies involved	$(q, \sigma(P_1), \sigma(P_2))$ • Fast via $\varphi \circ \psi \circ \widehat{\tau}$
Verification	• Recompute σ as a chain of ℓ-isogenies of known kernels • $\deg(\sigma) = \ell^e \simeq p^{15/4}$	• Compute F an ℓ^e-isogeny of dimension 4 • $\deg(\sigma) = \ell^e \simeq \sqrt{p}$

Thank you for listening.

Find our pre-print here: https://eprint.iacr.org/2023/436