Signing with higher dimensional isogenies

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Joint work with Antonin Leroux, Damien Robert and Benjamin Wesolowski
Acknowledgements to Luca De Feo

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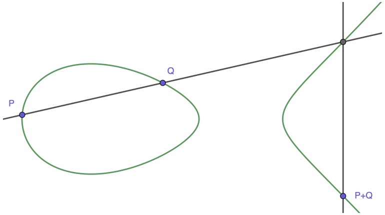
CANARI

Quaternions and isogenies

Elliptic curves



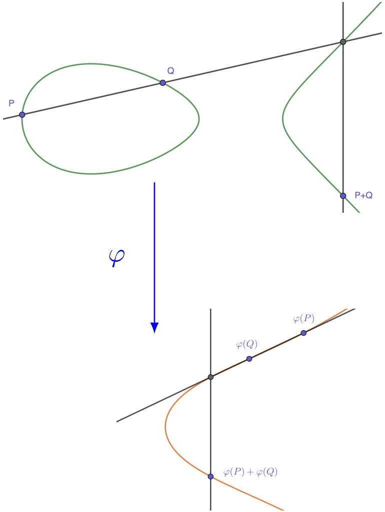
Elliptic curves:

- An elliptic curve $E/\mathbb{F}_q$ is defined by:

$$y^2 = x^3 + ax + b, \quad a, b \in \mathbb{F}_q$$

  with an infinite element 0_E.
- E is equipped with a commutative group law.

Isogenies - Definition



Isogenies:

Maps between elliptic curves $\varphi : E_1 \longrightarrow E_2$ such that:

- φ is a homomorphism of algebraic varieties.
- φ is a group homomorphism:

$$\varphi(P+Q) = \varphi(P) + \varphi(Q).$$

Isogenies - Examples

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  with $E/\mathbb{F}_{p^n}$ is an isogeny.
- An explicit example: Consider

$$E_1 : y^2 = x^3 + x + 4 \quad \text{and} \quad E_2 : y^2 = x^3 - x + 4$$

  over $\mathbb{F}_7$. Then

$$\begin{array}{rcl} \varphi : E_1 & \longrightarrow & E_2 \\ (x, y) & \longmapsto & \left(\dfrac{x^2 - 2x - 1}{x - 2}, y \dfrac{x^2 + 3x - 2}{(x - 2)^2} \right) \end{array}$$

  is an isogeny.

Isogenies - the degree

Definition (The degree)

- Measures the "size" of an isogeny. More precisely

$$\deg(\varphi) = [k(E_1) : \varphi^* k(E_2)] = \max(\deg(f), \deg(g))$$

  where $\varphi(x, y) = (f(x)/g(x), \cdots)$.
- Is multiplicative: $\deg(\varphi \circ \psi) = \deg(\varphi)\deg(\psi)$.

If $\deg(\varphi) = n$, we say that φ is an n-isogeny.

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Definition (Dual isogeny)

If $\varphi : E_1 \longrightarrow E_2$ the dual $\widehat{\varphi} : E_2 \longrightarrow E_1$ satisfies $\widehat{\varphi} \circ \varphi = [\deg(\varphi)]_{E_1}$ and $\deg(\varphi) = \deg(\widehat{\varphi})$.

Isogenies - Examples

- The scalar multiplication $[n] : E \longrightarrow E$ is an isogeny of **degree** n^2.
- The Frobenius:

$$\begin{array}{rcl} \pi_p : E & \longrightarrow & E^{(p)} \\ (x, y) & \longmapsto & (x^p, y^p) \end{array}$$

  with $E/\mathbb{F}_{p^n}$ is an **inseparable** p**-isogeny**.
- An explicit example: Consider

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  is a **separable** 2-**isogeny**.

The Endomorphism ring

Definition (Endomorphism ring)

$$\mathrm{End}(E) = \{0\} \cup \{\text{Isogenies } \varphi : E \longrightarrow E\}$$

Defines a ring for the addition and composition of isogenies.

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Theorem (Deuring)

Let $E/\mathbb{F}_q$ ($p = \mathrm{char}(\mathbb{F}_q)$). Then $\mathrm{End}(E)$ is either isomorphic to:

- *An order in a quadratic imaginary field. We say that E is ordinary.*
- *A maximal order in a quaternion algebra ramifying at p and ∞. We say that E is supersingular.*

If E is supersingular, we may assume $\mathbb{F}_q = \mathbb{F}_{p^2}$.

Quaternions - Definitions

- **Quaternion algebra ramifying at p and ∞:** A 4-dimensional non commutative division algebra over $\mathbb{Q}$:

$$\mathcal{B}_{p,\infty} = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}k,$$

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$$i^2 = -1 \text{ (if } p \equiv 3 \mod 4), \quad j^2 = -p \quad \text{and} \quad k = ij = -ji.$$

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- **Order:** A full rank lattice $\mathcal{O} \subset \mathcal{B}_{p,\infty}$ with a ring structure.
- **Maximal Order:** An order $\mathcal{O} \subset \mathcal{B}_{p,\infty}$ such that for any other order $\mathcal{O}' \supseteq \mathcal{O}$, we have $\mathcal{O}' = \mathcal{O}$.

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- **Left Ideal:** A left $\mathcal{O}$-ideal I is a full rank lattice $I \subset \mathcal{B}_{p,\infty}$ such that $\mathcal{O} \cdot I = I$.
- **Right Ideal:** A right $\mathcal{O}$-ideal I is a full rank lattice $I \subset \mathcal{B}_{p,\infty}$ such that $I \cdot \mathcal{O} = I$.

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- **Ideal conjugate:** $\overline{I} := \{\overline{\alpha} \mid \alpha \in I\}$.
- **Equivalent left $\mathcal{O}$-ideals:** $I \sim J \Longleftrightarrow \exists \alpha \in \mathcal{B}_{p,\infty}^*, \quad J = I\alpha$.

The Deuring correspondence

Supersingular elliptic curves	Quaternions
$j(E)$ or $j(E)^p$ supersingular	$\mathcal{O} \cong \mathrm{End}(E)$ maximal order in $\mathcal{B}_{p,\infty}$
$\varphi : E \longrightarrow E'$	left $\mathcal{O}$-ideal and right $\mathcal{O}'$-ideal I_φ
$\varphi, \psi : E \longrightarrow E'$	$I_\varphi \sim I_\psi$ $(I_\psi = I_\varphi \alpha)$
$\widehat{\varphi}$	$\overline{I_\varphi}$
$\varphi \circ \psi$	$I_\psi \cdot I_\varphi$
$\theta \in \mathrm{End}(E)$	Principal ideal $\mathcal{O}\theta$
$\deg(\varphi)$	$\mathrm{nrd}(I_\varphi)$

Computing isogenies (Easy)

Proposition

A separable isogeny is determined by its kernel. If $\varphi : E_1 \longrightarrow E_2$ *and* $\varphi' : E_1 \longrightarrow E_2'$ *are separable and* $\ker(\varphi) = \ker(\varphi')$, *then there exists an isomorphism* $\lambda : E_2 \xrightarrow{\sim} E_2'$ *such that* $\varphi' = \lambda \circ \varphi$.

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- Given $G = \ker(\varphi)$, we can compute φ in time $O(\sqrt{\#G})$ [BDLS20].

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- Given $G = \ker(\varphi)$, we can compute φ in time $O(\sqrt{\#G})$ [BDLS20].
- Given $P \in E$ such that $G = \langle P \rangle$ and if $\deg(\varphi) = \ell^n$ with ℓ small, we compute $\varphi : E \longrightarrow E'$ as a sequence of ℓ-isogenies

$$E = E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_{n-1} \longrightarrow E_n$$

in time $O(n \log(n))$ [JD11].

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- Only smooth degree isogenies can be computed efficiently.

The Supersingular Isogeny Problem (Hard)

Problem (Supersingular Isogeny Problem)

Given two supersingular elliptic curves $E_1, E_2/\mathbb{F}_{p^2}$, find an isogeny $\varphi : E_1 \longrightarrow E_2$.

When p has cryptographic size, this problem is hard for quantum computers.

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An easy instance: Consider $E_0 : y^2 = x^3 + x$ over $\mathbb{F}_p$ ($p \equiv 3 \mod 4$) and

$$\pi_p : (x, y) \in E_0 \longmapsto (x^p, y^p) \in E_0$$

$$\iota : (x, y) \in E_0 \longmapsto (x, \sqrt{-1}y) \in E_0$$

Then $\pi_p^2 = [-p]$, $\iota^2 = [-1]$ and $\pi_p \circ \iota = -\iota \circ \pi_p$ and

$$\mathrm{End}(E_0) = \left\langle 1, \iota, \frac{\iota + \pi_p}{2}, \frac{1 + \iota\pi_p}{2} \right\rangle \cong \left\langle 1, i, \frac{i + j}{2}, \frac{1 + k}{2} \right\rangle$$

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Theorem (Wesolowski, 2022)

The Supersingular Isogeny Problem and the Supersingular Endomorphism Ring Problem are equivalent.

Quaternion path problem (Easy)

Problem (Connecting ideal)

Given two maximal orders $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{B}_{p,\infty}$, find a left $\mathcal{O}_1$-ideal I that is also a right $\mathcal{O}_2$-ideal.

This is simple arithmetic $I \sim \mathcal{O}_1 \cdot \mathcal{O}_2$.

Problem (Quaternion path problem)

Given a left $\mathcal{O}$-ideal I, find $J \sim I$ of smooth norm.

Solved in polynomial time by the KLPT algorithm [KLPT14; DKLPW20].

Quaternion path problem (Easy)

Computing isogenies via the Deuring correspondence:

- Let E_1 and E_2 of known endomorphism rings $\mathcal{O}_1 \cong \mathrm{End}(E_1)$ and $\mathcal{O}_2 \cong \mathrm{End}(E_2)$.
- Compute a connecting ideal I between $\mathcal{O}_1$ and $\mathcal{O}_2$.
- Compute $J \sim I$ of smooth norm via KLPT.
- Translate J into an isogeny $\varphi_J : E_1 \longrightarrow E_2$.

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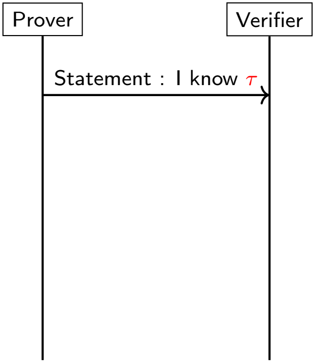
Becomes hard when $\text{End}(E_1)$ or $\text{End}(E_2)$ is unknown.

SQISign and effective Deuring correspondence

The SQISign identification scheme [DKLPW20; DLW22]

$$E_0 \xrightarrow{\tau} E_A$$



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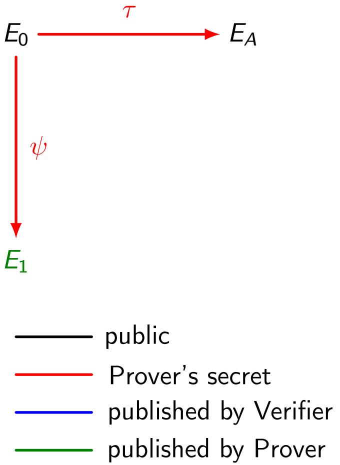

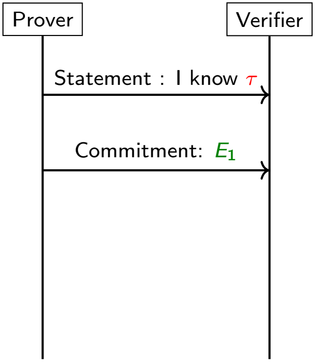

The SQISign identification scheme [DKLPW20; DLW22]

$$\begin{tikzcd} E_0 \arrow[r, "\tau"] \arrow[d, "\psi"] & E_A \\ E_1 \arrow[r, "\varphi"] & E_2 \end{tikzcd}$$

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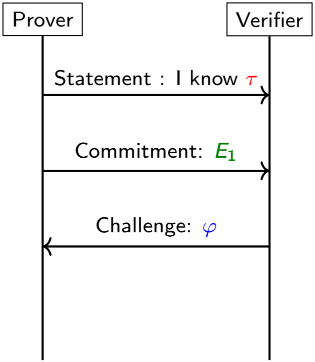

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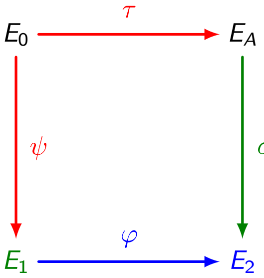

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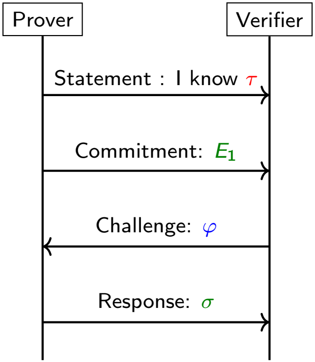

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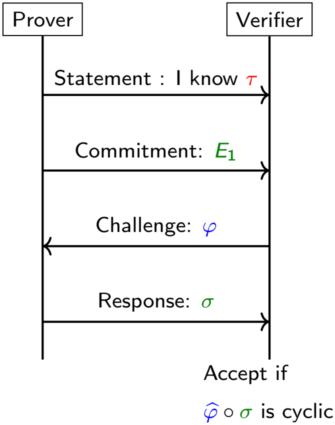

Fiat-Shamir transform [FS87]

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Signature: message m, public key E_A, secret key τ.

- Commitment $\psi : E_0 \longrightarrow E_1$.
- Challenge $\varphi := H(E_1, m)$ (where H is a hash function).
- Compute and send signature (E_1, σ) to the verifier.

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Verification: $E_A, m, (E_1, \sigma)$.

- Recompute $\varphi := H(E_1, m)$.
- Verify that $\widehat{\varphi} \circ \sigma$ is cyclic.

How to compute the signature σ ?

$$
\begin{tikzcd}
E_0 \arrow[r, "\tau", "I_\tau"'] \arrow[d, "\psi", "I_\psi"'] & E_A \arrow[d, dashed, "\sigma", "I"'] \\
E_1 \arrow[r, "\varphi", "I_\varphi"'] & E_2
\end{tikzcd}
$$

- Compute $J := \overline{I_\tau} \cdot I_\psi \cdot I_\varphi$.
- Find $I \sim J$ random of norm $\mathrm{nrd}(I) = \ell^e$ (KLPT).
- Compute σ associated to I.

Ideal-to-isogeny I [GPS16]

Input: $E/\mathbb{F}_{p^2}$ supersingular, $\mathcal{O} \cong \mathrm{End}(E)$ and I a left $\mathcal{O}$-ideal of smooth norm.

Output: $\varphi_I : E \longrightarrow E_I$.

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Issue: If I is a KLPT output, then $\mathrm{nrd}(I) \simeq p^{15/4} \gg p$ so k is exponentially big. Not practical for SQISign !

Ideal-to-isogeny II [DLW22]

Main idea: Cut the computation into smaller pieces. Write

$$I = I_0 \cdot I_1 \cdots I_{n-1} \quad \text{and} \quad \varphi_I = \varphi_{n-1} \circ \cdots \circ \varphi_1 \circ \varphi_0$$

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E = E_0 \arrow[r, "I_0", "\varphi_0"'] & E_1 \arrow[loop above, "\theta_1"] \arrow[r, "I_1", "\varphi_1"'] & E_2 \arrow[loop above, "\theta_2"] & \cdots & E_{n-1} \arrow[loop above, "\theta_{n-1}"] \arrow[r, "I_{n-1}", "\varphi_{n-1}"'] & E_n = E_I
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The endomorphisms θ_i are meant to refresh the ℓ^f-torsion.

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Main idea: Cut the computation into smaller pieces. Write

$$I = I_0 \cdot I_1 \cdots I_{n-1} \quad \text{and} \quad \varphi_I = \varphi_{n-1} \circ \cdots \circ \varphi_1 \circ \varphi_0$$

with $\mathrm{nrd}(I_0) = \cdots = \mathrm{nrd}(I_{n-1}) = \ell^f$.

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E = E_0 \arrow[r, "I_0", "\varphi_0"'] & E_1 \arrow[loop above, "\theta_1"] \arrow[r, "I_1", "\varphi_1"'] & E_2 \arrow[loop above, "\theta_2"] & \cdots & E_{n-1} \arrow[loop above, "\theta_{n-1}"] \arrow[r, "I_{n-1}", "\varphi_{n-1}"'] & E_n = E_I
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Issue: This is slow!

Higher dimensional isogenies

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Theorem (Robert, 2022)

Let $\sigma : E \longrightarrow E'$ of degree $q < \ell^e$. There exists a polynomial time algorithm with:

- ***Input:*** $(\sigma(P_1), \sigma(P_2))$, *where* (P_1, P_2) *is a basis of* $E[\ell^e]$ *and* $Q \in E(\mathbb{F}_{p^2})$.
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Context: This idea comes from the attacks against SIDH [CD22; MM22; Rob22].

Evaluating σ

Main idea: Use the alternate path $\varphi \circ \psi \circ \widehat{\tau}$.

$$\begin{tikzcd}
E_0 \arrow[r, "\tau", "I_\tau"'] \arrow[d, "\psi"', "I_\psi"] & E_A \arrow[d, dashed, "\sigma", "I"'] \\
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- Let $\gamma := \widehat{\psi} \circ \widehat{\varphi} \circ \sigma \circ \tau \in \mathrm{End}(E_0)$.
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- Then:

$$[D_\psi D_\varphi D_\tau]\sigma = \varphi \circ \psi \circ \gamma \circ \widehat{\tau}$$

- We can evaluate σ on $P \in E_A[\ell^e]$ provided $(D_\psi D_\varphi D_\tau) \wedge \ell = 1$:

$$\sigma(P) = [\lambda]\varphi \circ \psi \circ \gamma \circ \widehat{\tau}(P),$$

with $\lambda D_\psi D_\varphi D_\tau \equiv 1 \mod \ell^e$.

d-isogenies and the dual isogeny in higher dimension

Definition (d-isogeny)

Let $\varphi : (A, \lambda_A) \longrightarrow (B, \lambda_B)$ be an isogeny between two principally polarized abelian varieties (PPAV). We define:

- $\widetilde{\varphi} := \lambda_A^{-1} \circ \widehat{\varphi} \circ \lambda_B : B \longrightarrow A$.

$$B \xrightarrow{\lambda_B} \widehat{B} \xrightarrow{\widehat{\varphi}} \widehat{A} \xrightarrow{\lambda_A^{-1}} A$$

- We say that φ is a <u>d-isogeny</u> if $\widetilde{\varphi} \circ \varphi = [d]_A$.

Kani's embedding lemma [Kan97]

Definition (isogeny diamond)

An (a, b)-isogeny diamond is a commutative diagram s.t.:

$$\begin{tikzcd} A' \arrow[r, "\varphi'"] & B' \\ A \arrow[u, "\psi"] \arrow[r, "\varphi"] & B \arrow[u, "\psi'"'] \end{tikzcd}$$

where φ, φ' are a-isogenies and ψ, ψ' are b-isogenies.

Lemma (Kani)

Consider the (a, b)-isogeny diamond on the left. Then:

- $F : A \times B' \longrightarrow B \times A'$,

  $$F := \begin{pmatrix} \varphi & \widetilde{\psi}' \\ -\psi & \widetilde{\varphi}' \end{pmatrix}$$

  is a d-isogeny with $d = a + b$.
- *If $a \wedge b = 1$, then*

  $$\ker(F) = \{(\widetilde{\varphi}(x), \psi'(x)) \mid x \in B[d]\}.$$

Application of Kani's lemma to SQISignHD

Embedding σ in higher dimension:

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- Let $q = \deg(\sigma)$.
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- q should be good: $\ell^e - q$ prime $\equiv 1 \mod 4$.

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- q should be good: $\ell^e - q$ prime $\equiv 1 \mod 4$.
- Consider the isogeny diamond:

$$\begin{tikzcd} E_2^2 \arrow[r, "\alpha_2"] & E_2^2 \\ E_A^2 \arrow[u, "\Sigma"] \arrow[r, "\alpha_A"] & E_A^2 \arrow[u, "\Sigma"'] \end{tikzcd}$$

where $\Sigma := \mathrm{Diag}(\sigma, \sigma)$ and for $i = A, 2$,

$$\alpha_i := \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix} \in \mathrm{End}(E_i^2).$$

Application of Kani's lemma to SQISignHD

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$$\ker(F) = \{([a_1]R - [a_2]S, [a_2]R + [a_1]S, \sigma(R), \sigma(S)) \mid R, S \in E_A[\ell^e]\}.$$

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$$\ker(F) = \{([a_1]R - [a_2]S, [a_2]R + [a_1]S, \sigma(R), \sigma(S)) \mid R, S \in E_A[\ell^e]\}.$$
- F can be computed in polynomial time [LR12; LR15; LR23; DLRW23].

Algorithm for higher dimensional isogeny computations

- The ℓ^e-isogeny F can be computed as a chain of ℓ-isogenies:

$$\mathcal{A}_0 \xrightarrow{F_0} \mathcal{A}_1 \xrightarrow{F_2} \mathcal{A}_2 \quad \cdots \quad \mathcal{A}_{e-1} \xrightarrow{F_e} \mathcal{A}_e$$

- Each ℓ-isogeny can be computed in $O(\ell^g)$ efficiently in the Θ-model [LR12; LR15; LR23; DLRW23].
- The whole chain can be computed in time $O(\ell^g e \log(e))$ [JD11; DLRW23].
- This method is valid in any dimension g.

Concrete implementation in dimension 4

- We provide a `sagemath` implementation when $\ell = 2$ with Θ-coordinates of level 2 (16 coordinates).

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- Concrete test with $q = 3^{79}$ and $e = 128$.
- Runs in 600 ms on a 13th Gen Intel(R) Core(TM) i5-1335U (4600MHz) CPU.

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Precomputation time: 0.049009084701538086 s
Computation time: 0.616366863250 7324 s
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- Target after optimizations: gain a factor 10-50.

SQISignHD: signing with higher dimensional isogenies

SQISignHD identification scheme [DLRW23]

$$\begin{array}{ccc} E_0 & \xrightarrow{\tau} & E_A \\ \Big\downarrow \psi & & \Big\downarrow \sigma \\ E_1 & \xrightarrow{\varphi} & E_2 \end{array}$$

Secret key: τ

Commitment: E_1

Challenge: φ

public

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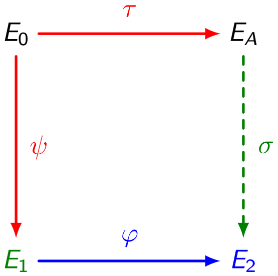

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Very fast !

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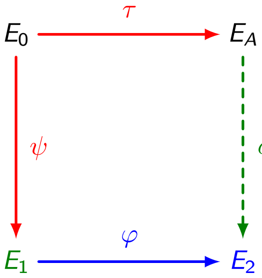

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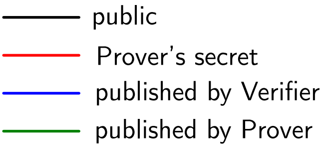

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- Compute F.
- Accept if $F \in \mathrm{End}(E_A^2 \times E_2^2)$ and $F(Q, 0, 0, 0) = ([a_1]Q, -[a_2]Q, *, 0)$.

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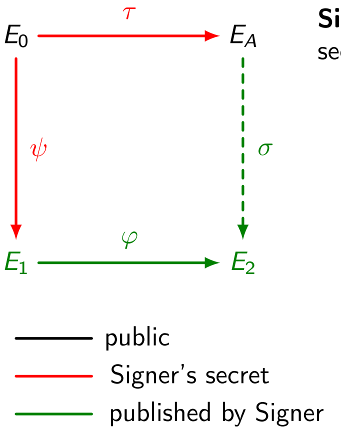
Implementation in progress.

Fiat-Shamir transform [FS87] of SQISignHD



Signature: message m, public key E_A, secret key τ.

- Commitment $\psi : E_0 \longrightarrow E_1$.
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public

Signer's secret

published by Signer

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Verification: $E_A, m, (E_1, q, \sigma(P_1), \sigma(P_2))$.

- Recompute $\varphi := H(E_1, m)$.
- Use $\varphi, q, \sigma(P_1), \sigma(P_2)$ to compute the embedding F of σ.
- Check that $F \in \mathrm{End}(E_A^2 \times E_2^2)$ and $F(Q, 0, 0, 0) = ([a_1]Q, -[a_2]Q, *, 0)$.

Outline of the security analysis

Theorem (Fiat-Shamir, 1986)

Let ID be an identification protocol that is:

- ***Complete:*** *a honest execution is always accepted by the verifier.*
- ***Sound:*** *an attacker cannot "guess" a response.*
- ***Zero-knowledge:*** *the response does not leak any information on the secret key.*

Then the Fiat-Shamir transform of ID is a universally unforgeable signature under chosen message attacks in the random oracle model.

Soundness

Similar to SQISign

Proposition (Special soundness)

Given two transcripts $(E_1, \varphi, q, \sigma(P_1), \sigma(P_2)), (E_1, \varphi', q', \sigma'(P_1'), \sigma'(P_2'))$ *with the same commitment* E_1 *and* $\varphi \neq \varphi'$, *we can extract* $\alpha \in \mathrm{End}(E_A)$ *non-scalar.*

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Proof.

- Exctract σ from $(q, \sigma(P_1), \sigma(P_2))$ and σ' from $(q', \sigma'(P_1'), \sigma'(P_2'))$.
- Then $\alpha := \widehat{\varphi}' \circ \widehat{\sigma}' \circ \sigma \circ \varphi \in \mathrm{End}(E_A)$ is non-scalar.

$\square$

Zero-knowledge

Definition (Recall)

We say that an integer q is **good** if $\ell^e - q$ is a prime $\equiv 1 \mod 4$.

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Definition (RUGDIO)

A random uniform good degree isogeny oracle (RUGDIO):
Input: A supersingular elliptic curve $E/\mathbb{F}_{p^2}$.
Output: An isogeny $\sigma : E \longrightarrow E'$ of good degree q s.t.

- E' is uniform among supersingular elliptic curves.
- Given E', σ is uniform among isogenies of good degree $E \longrightarrow E'$.

Zero-knowledge

Theorem

Assume that:

- *E_1 is computationally close to uniform.*
- *We have access to a RUGDIO.*

Then SQISignHD is computationally honest-verifier zero-knowledge.

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Proof.

We build a simulator $\mathcal{S}$ of protocol transcripts:

- $\mathcal{S}$ calls the RUGDIO to generate $(q, \sigma(P_1), \sigma(P_2))$.
- $\mathcal{S}$ generates a random challenge $\widehat{\varphi} : E_2 \longrightarrow E_1$.
- $\mathcal{S}$ outputs $(E_1, \varphi, q, \sigma(P_1), \sigma(P_2))$.

$\square$

Zero-knowledge: comparison with SQISign

Heuristic assumptions to prove the zero-knowledge property

In SQISign:

- $\sigma : E_A \longrightarrow E_2$ is computationally indistinguishable from a random isogeny of degree ℓ^e.

In SQISignHD:

- E_1 is computationally close to uniform.
- We have access to a RUGDIO.

Compact signatures

Signature size comparison

	In SQISign	In SQISignHD
Asymptotic (in bits)	$\sim 23/4 \log_2(p)$	$\sim 13/4 \log_2(p)$
NIST-1 security level (in bytes)	204	116

Conclusion

Comparison of SQISignHD with SQISign

	SQISign	SQISignHD
Security	✗ Ad-hoc heuristic: • Distribution of σ.	✓ Simpler heuristics: • RUGDIO; • Distribution of E_1.
Signing time	✗ 400 ms for NIST-1	✓ $<$ 60 ms for NIST-1
Signature size	✓ 204 bytes for NIST-1	✓ 116 bytes for NIST-1
Verification	✓ Fast (6 ms for NIST-1)	✗ 600 ms for NIST-1 in `sagemath`

Thank you for listening.

Find our pre-print here: https://eprint.iacr.org/2023/436